

ESTIMATES OF THE DERIVATIVES FOR PARABOLIC OPERATORS WITH UNBOUNDED COEFFICIENTS

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ABSTRACT. We consider a class of second-order uniformly elliptic operators \mathcal{A} with unbounded coefficients in \mathbb{R}^N . Using a Bernstein approach we provide several uniform estimates for the semigroup $T(t)$ generated by the realization of the operator \mathcal{A} in the space of all bounded and continuous or Hölder continuous functions in \mathbb{R}^N . As a consequence, we obtain optimal Schauder estimates for the solution to both the elliptic equation $\lambda u - \mathcal{A}u = f$ ($\lambda > 0$) and the nonhomogeneous Dirichlet Cauchy problem $D_t u = \mathcal{A}u + g$. Then, we prove two different kinds of pointwise estimates of $T(t)$ that can be used to prove a Liouville-type theorem. Finally, we provide sharp estimates of the semigroup $T(t)$ in weighted L^p -spaces related to the invariant measure associated with the semigroup.

1. INTRODUCTION

In recent years much attention has been paid to the uniformly elliptic operator

$$\mathcal{A} = \sum_{i,j=1}^N q_{ij}(x)D_{ij} + \sum_{j=1}^N b_j(x)D_j,$$

with unbounded coefficients in \mathbb{R}^N . The interest for operators with unbounded coefficients is due to the fact that they arise naturally in the theory of Markov processes. Actually these operators have very interesting properties, quite different from those enjoyed by elliptic operators with bounded coefficients.

If f is continuous and bounded in \mathbb{R}^N ($f \in C_b(\mathbb{R}^N)$ for short), under suitable hypotheses the Cauchy problem

$$(1.1) \quad \begin{cases} D_t u(t, x) = \mathcal{A}u(t, x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0, x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

has a unique bounded solution. Moreover, there exists a semigroup $\{T(t)\}_{t \geq 0}$ of linear operators in $C_b(\mathbb{R}^N)$ such that $u(t, x) = (T(t)f)(x)$. In general, $T(t)$ is neither strongly continuous nor analytic (see [14]).

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The problem of estimating the derivatives of $T(t)f$ has already been studied in literature by both analytic ([2, 4, 6, 12]) and probabilistic methods ([5, 19]).

In this paper we prove both uniform and pointwise estimates for the first-, second- and third-order derivatives of $T(t)f$. In section 3 we prove uniform estimates, namely we show that for any $\omega > 0$ and any $k, l \in \mathbb{N}$, $0 \leq k \leq l \leq 3$, there exists a positive constant $C_{k,l} = C_{k,l}(\omega)$ such that

$$(1.2) \quad \|T(t)f\|_{C_b^l(\mathbb{R}^N)} \leq C_{k,l} t^{-(l-k)/2} e^{\omega t} \|f\|_{C_b^k(\mathbb{R}^N)}, \quad \forall f \in C_b^k(\mathbb{R}^N), \quad t > 0.$$

To prove (1.2) we use the Bernstein method and approximate $T(t)f$ by solutions of Cauchy problems in bounded domains. We assume dissipativity-type and growth conditions on the coefficients of \mathcal{A} . We notice that some dissipativity condition is necessary, because in general estimate (1.2) fails; see [4].

Then, using interpolation arguments, we obtain similar estimates for $T(t)f$ in Hölder spaces. Namely, we extend estimate (1.2) to the case when $k, l \in \mathbb{R}_+$, $0 \leq k \leq l \leq 3$. This allows us to prove optimal Schauder estimates for the solution of the nonhomogeneous Dirichlet Cauchy problem associated with (1.1) as well as for the elliptic equation $\lambda u - \mathcal{A}u = f$ ($\lambda > 0$).

In section 4 we consider the following pointwise estimates for the derivatives of $T(t)f$: first we prove that for any $k = 1, 2, 3$ and any $p \in (1, +\infty)$, there exists a constant $\sigma_{k,p} \in \mathbb{R}$ such that

$$(1.3) \quad |(D^k T(t)f)(x)|^p \leq e^{\sigma_{k,p} t} \left(T(t)((|f|^2 + \dots + |D^k f|^2)^{\frac{p}{2}}) \right)(x), \quad \forall t > 0, \quad x \in \mathbb{R}^N,$$

for all $f \in C_b^k(\mathbb{R}^N)$. Consequently we deduce the second pointwise estimate

$$(1.4) \quad |(D^k T(t)f)(x)|^p \leq \left(\frac{\sigma_{k,\min\{p,2\}}}{1 - e^{-\sigma_{k,\min\{p,2\}} t}} \phi_{k,\min\{p,2\}}(t) \right)^{\max\{1, \frac{p}{2}\}} \times \left(T(t)((|f|^2 + \dots + |D^{k-1} f|^2)^{\frac{p}{2}}) \right)(x),$$

for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^N$, any $f \in C_b^{k-1}(\mathbb{R}^N)$ and any $p > 1$, where $\phi_{k,r} \in C([0, +\infty))$ is a suitable function which behaves as $t^{1-r/2}$ near 0 and may grow exponentially at infinity (see (4.16)). Then, taking the semigroup property into account, from (1.4) one readily obtains the estimate

$$(1.5) \quad |(D^k T(t)f)(x)|^p \leq C_{k,p} \frac{e^{\omega_{k,p} t}}{t^{pk/2}} (T(t)|f|^p)(x), \quad t > 0, \quad x \in \mathbb{R}^N,$$

for any $f \in C_b(\mathbb{R}^N)$, any $p > 1$ and some constants $C_{k,p} > 0$, blowing up as p tends to 1, and $\omega_{k,p} \in \mathbb{R}$.

In the particular case when $q_{ij}(x) = \delta_{ij}$, i.e. when $\mathcal{A} = \Delta + \sum b_i(x) D_i$, we also prove estimate (1.3) for $p = 1$. On the contrary, estimate (1.5) cannot, in general, be extended to the case when $p = 1$. Counterexamples are easily obtained in the simple case $\mathcal{A} = \Delta$ (see [14]).

Such pointwise estimates are typical for transition semigroups of Markov processes, and they have already been studied for the first-order derivatives ($k = 1$); see [1, 2, 3].

In the case when $\omega_{1,p} \leq 0$, estimate (1.5) with $k = 1$ allows us to obtain a Liouville-type theorem, namely, in such a situation we can show that, if $\mathcal{A}u = 0$, then u is constant. If $\omega_{1,p} > 0$, in general, such a result fails. Counterexamples are also given in [18] in the one-dimensional case.

In the last section we derive from (1.5) similar estimates for the derivatives of the extension of $T(t)f$ to the L^p -spaces related to the invariant measure μ associated with the semigroup. To be more precise, we show that, for any $\omega > 0$, any $k = 1, 2, 3$ and any $p \in (1, +\infty)$, there exists a positive constant $C_{k,p} = C_{k,p}(\omega)$ such that

$$(1.6) \quad \|D^k T(t)f\|_p \leq C_{k,p} \frac{e^{\omega_{k,p}t}}{t^{k/2}} \|f\|_p, \quad t > 0,$$

for all $f \in L^p(\mathbb{R}^N, \mu)$, where $\|\cdot\|_p$ denotes the norm of $L^p(\mathbb{R}^N, \mu)$.

The characterization of the domain $D(A_p)$ of the realization of A_p in $L^p(\mathbb{R}^N, \mu)$ is still an open problem. There are just a few results in the particular case when the drift term is a gradient (see [16]) or \mathcal{A} is an Ornstein-Uhlenbeck operator (see [15]). In such a situation, the invariant measure can be explicitly determined, and things are easier. In general, the invariant measure is unknown and there are only a few results on the regularity and behaviour at infinity of its density $d\mu/dx$, which do not allow us to perform the same technique as in [16] to characterize $D(A_p)$.

One of the main consequence of (1.6), with $k = 1$, is that it allows us to give a partial characterization of $D(A_p)$. Indeed, using a standard technique relying on the representation of the resolvent $R(\lambda, A_p)f$, at $x \in \mathbb{R}^N$, as the Laplace transform of the function $t \mapsto (T(t)f)(x)$, we show that if $f \in D(A_p)$, then the first distributional derivatives $D_k f$, $k = 1, \dots, N$, belong to $L^p(\mathbb{R}^N, \mu)$.

Notation. Throughout the paper we denote by $B(R)$ the open ball in \mathbb{R}^N with center at $x = 0$ and radius R , and by $\overline{B}(R)$ its closure in \mathbb{R}^N . For any $a, b \in \mathbb{R}$, we denote by $a \vee b$ and $a \wedge b$, respectively, the maximum and the minimum between a and b . When $b = 0$ we set $a^+ = a \vee 0$. For any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, any open set $\Omega \subset \mathbb{R}^N$ and any smooth function $f : \Omega \rightarrow \mathbb{R}$ we set $D^\alpha f = D_{x_1}^{\alpha_1} \dots D_{x_N}^{\alpha_N} f$. Similarly, for any $k \in \mathbb{N}$ and any $x \in \Omega$ we set $|D^k f(x)|^2 = \sum_{|\alpha|=k} |D^\alpha f(x)|^2$.

2. FUNCTION SPACES AND PRELIMINARIES

In this section we define the function spaces we deal with throughout this paper and we collect all the needed preliminary results.

Definition 2.1. We denote by $C_b(\mathbb{R}^N)$ the space of all bounded and continuous functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ endowed with the sup-norm. Moreover, for any $k \in \mathbb{R}_+$, we denote by $C_b^k(\mathbb{R}^N)$ the spaces of all continuously differentiable up to the $[k]$ -order functions f such that $D^\alpha f \in C_b(\mathbb{R}^N)$ for any $|\alpha| \leq [k]$ ($[k]$ denoting the integer part of k) and $D^\alpha f$ is $(k - [k])$ -Hölder continuous. We endow $C_b^k(\mathbb{R}^N)$ with the norm

$$\|f\|_{C_b^k(\mathbb{R}^N)} = \sum_{|\alpha| \leq [k]} \|D^\alpha f\|_\infty + \sum_{|\alpha| = [k]} [D^\alpha f]_{C^{k-[k]}(\mathbb{R}^N)}, \quad \forall f \in C_b^k(\mathbb{R}^N),$$

where

$$[g]_{C^{k-[k]}(\mathbb{R}^N)} = \sup_{x, y \in \mathbb{R}^N, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^{k-[k]}}, \quad \forall g \in C^{k-[k]}(\mathbb{R}^N).$$

For any open set $\Omega \subset \mathbb{R}^N$ and any $k \in [0, +\infty]$, $C_0^k(\Omega)$ denotes the subset of $C^k(\mathbb{R}^N)$ of all the functions f compactly supported in Ω .

By $C_{loc}^k(\mathbb{R}^N)$, $k \in \mathbb{R}_+$, we denote the set of all the functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ which belong to $C^k(K)$ for any compact set $K \subset \mathbb{R}^N$. Similarly,

$$C_{loc}^{1+\alpha/2, 2+\alpha}((0, +\infty) \times \mathbb{R}^N)$$

denotes the space of the functions u which are once continuously differentiable with respect to time and twice continuously differentiable with respect to the space variables in $(0, +\infty) \times \mathbb{R}^N$ and such that for any compact set $D \subset (0, +\infty) \times \mathbb{R}^N$, $D_t u, D_x^\beta u$ ($|\beta| \leq 2$) are α -Hölder continuous in D with respect to the parabolic distance $d((t, x), (s, y)) = (|t - s| + |x - y|^2)^{1/2}$. For any $k \in \mathbb{N}$ and any compact set $D \subset \mathbb{R} \times \mathbb{R}^N$, $C^{1+\alpha/2, 2+k+\alpha}(D)$ denotes the set of all the u 's which admit space derivatives up to the k -th order in $C^{1+\alpha/2, 2+\alpha}(D)$.

For any positive measure μ we denote by $L^p(\mathbb{R}^N, \mu)$ ($p \in [1, +\infty)$) (L_μ^p for short) the L^p -space related to the measure μ and we endow it with the usual norm.

Finally, we denote by $W^{k,p}(\mathbb{R}^N, \mu)$ ($k \in \mathbb{N}$, $p \in [1, +\infty)$) ($W_\mu^{k,p}$ for short) the space of all the functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ admitting weak derivatives up to the k -order in L_μ^p . We endow it with the norm

$$\|f\|_{W_\mu^{k,p}} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L_\mu^p}, \quad \forall f \in W_\mu^{k,p}.$$

We now recall some basic results on the Cauchy problem

$$(2.1) \quad \begin{cases} D_t u(t, x) = \mathcal{A}u(t, x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0, x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

where $f \in C_b(\mathbb{R}^N)$. Under the following hypothesis:

H1): $q_{ij}, b_j \in C_{loc}^\delta(\mathbb{R}^N)$ for some $\delta \in (0, 1)$ and $q_{ij}(x) = q_{ji}(x)$ for any $i, j = 1, \dots, N$ and any $x \in \mathbb{R}^N$, and

$$\sum_{i,j=1}^N q_{ij}(x) \xi_i \xi_j \geq \nu(x) |\xi|^2, \quad \forall \xi, x \in \mathbb{R}^N,$$

for some function $\nu : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\inf_{x \in \mathbb{R}^N} \nu(x) = \nu_0 > 0$,

problem (2.1) admits a classical solution

$$u \in C([0, +\infty) \times \mathbb{R}^N) \cap C_{loc}^{1+\delta/2, 2+\delta}((0, +\infty) \times \mathbb{R}^N)$$

satisfying

$$|u(t, x)| \leq \|f\|_\infty, \quad t > 0, \quad x \in \mathbb{R}^N.$$

Without any additional assumption, in general the function u is not the unique classical bounded solution to problem (2.1) (see [9, 13] for examples of nonuniqueness). If we assume the further condition

H2): there exists $\lambda > 0$ and a function $\varphi \in C^2(\mathbb{R}^N)$ such that

$$\lim_{|x| \rightarrow +\infty} \varphi(x) = +\infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^N} (\mathcal{A}\varphi(x) - \lambda\varphi(x)) < +\infty,$$

then the classical bounded solution to (2.1) is unique and is given by

$$(2.2) \quad u(t, x) = \int_{\mathbb{R}^N} G(t, x, y) f(y) dy, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N,$$

where G is a positive function such that $G(\cdot, \cdot, y) \in C_{loc}^{1+\delta/2, 2+\delta}((0, +\infty) \times \mathbb{R}^N)$, it solves the equation $D_t G(\cdot, \cdot, y) = \mathcal{A}G(\cdot, \cdot, y)$ for almost any $y \in \mathbb{R}^N$ and

$$(2.3) \quad \int_{\mathbb{R}^N} G(t, x, y) dy = 1, \quad \forall t > 0, \quad x \in \mathbb{R}^N.$$

The function G is called the fundamental solution to problem (2.1).

The family of bounded operators $\{T(t)\}_{t \geq 0}$ defined by $T(t)f = u(t, \cdot)$ for any $f \in C_b(\mathbb{R}^N)$ and any $t > 0$, gives rise to a contractive semigroup of linear operators in $C_b(\mathbb{R}^N)$. In general, $T(t)$ is neither strongly continuous nor analytic in $C_b(\mathbb{R}^N)$. Moreover, the following property holds:

$$(2.4) \quad \begin{aligned} & \text{If } \{f_n\}_{n \in \mathbb{N}} \subset C_b(\mathbb{R}^N) \text{ is a bounded sequence such that } \lim_{n \rightarrow +\infty} f_n = \\ & f \in C_b(\mathbb{R}^N) \text{ uniformly in } B(k) \text{ for all } k > 0, \text{ then } \lim_{n \rightarrow +\infty} (T(t)f_n)(x) = \\ & (T(t)f)(x) \text{ uniformly in } [0, T] \times B(k) \text{ for all } T, k > 0. \end{aligned}$$

We refer the reader to [3, 8, 13] for the proofs of the previous results.

The solution $u(t, x) = (T(t)f)(x)$ can be approximated with the solutions of Dirichlet problems. To be more precise let u_R be the solution to the Dirichlet Cauchy problem

$$(2.5) \quad \begin{cases} D_t u_R(t, x) = \mathcal{A}u_R(t, x), & t > 0, \quad x \in B(R), \\ u_R(t, x) = 0, & t > 0, \quad x \in \partial B(R), \\ u_R(0, x) = \eta_R(x)f(x), & x \in B(R), \end{cases}$$

where $f \in C(\overline{B}(R))$, and η_R is any nonnegative smooth function compactly supported in $B(R)$ and such that $\eta_R \equiv 1$ in $B(R/2)$.

We denote by $\{T_R(t)\}_{t \geq 0}$ the semigroup in $C(\overline{B}(R))$ associated with problem (2.5), so that $u_R(t, \cdot) = T_R(t)(\vartheta_R f)$; the generator of $T_R(t)$ is the operator $A : D(A_R) \rightarrow X$ defined by

$$(2.6) \quad D(A_R) = \left\{ u \in C(\overline{B}(R)) \cap \bigcap_{1 < p < +\infty} W_{loc}^{2,p}(B(R)) : u|_{\partial B(R)} = 0, \mathcal{A}u \in C(\overline{B}(R)) \right\},$$

and $A_R u = \mathcal{A}u$ for any $u \in D(A_R)$; see [11, Corollary 3.1.21].

Now let $f \in C_b(\mathbb{R}^N)$ and fix $\varepsilon, T, n > 0$. Then we have

$$(2.7) \quad \lim_{R \rightarrow \infty} \|u - u_R\|_{C^{1+\delta/2, 2+\delta}([\varepsilon, T] \times B(n))} = 0;$$

see e.g. [3, Theorem 1.12], [13, Theorem 4.2]. Moreover, the fundamental solution G_R to problem (2.5) is positive for any $R > 0$, $G_{R_1} < G_{R_2}$ for all $R_1 < R_2$, and G_R tends to the function G in (2.2) as R tends to $+\infty$. As straightforward consequences, we deduce that, for any $f \geq 0$ and $R_1 < R_2$, we have

$$(2.8) \quad T_{R_1}(t)f \leq T_{R_2}(t)f \leq T(t)f, \quad \forall t > 0$$

(see also [13, Lemma 4.1]), and, for any $f \in C_b(\mathbb{R}^N)$,

$$(2.9) \quad |T(t)f - T_R(t)f| \leq (T(t) - T_R(t))|f|, \quad \forall t > 0.$$

Let us now recall the following classical interior estimate (see e.g. [7, Theorems 3.5 and 3.10]).

Theorem 2.2. *Suppose that $q_{ij}, b_j \in C_{loc}^{k+\delta}(\mathbb{R}^N)$ for any $i, j = 1, \dots, N$, for some $k \in \mathbb{N}$ and $\delta \in (0, 1)$, and suppose that H1 holds. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let $u \in L^\infty((0, T) \times \Omega)$ be a solution of the equation $u_t = \mathcal{A}u$ in $(0, T) \times \Omega$.*

Then, for any $|\beta| \leq k$, the function $D^\beta u$ belongs to $C_{loc}^{1+\delta/2, 2+\delta}((0, +\infty) \times \Omega)$. Moreover, for any $\varepsilon, \varepsilon' > 0$ ($\varepsilon < \varepsilon' < T$), and any open set $\Omega' \subset \subset \Omega$,

there exists a positive constant C depending on $\varepsilon, \varepsilon', T, \Omega, \Omega', k, \|q_{ij}\|_{C^{k+\delta}(\Omega)}$ and $\|b_j\|_{C^{k+\delta}(\Omega)}$ ($i, j = 1, \dots, N$), such that

$$(2.10) \quad \|u\|_{C^{1+\delta/2, k+2+\delta}([\varepsilon', T] \times \Omega')} \leq C \sup_{(t, x) \in [\varepsilon, T] \times \Omega} |u(t, x)|.$$

Theorem 2.2 yields the following consequence: applying (2.10) to $u - u_R$, by (2.7) it follows that

$$(2.11) \quad \lim_{R \rightarrow +\infty} \|u_R - u\|_{C^{1+\delta/2, k+2+\delta}([\varepsilon, T] \times B(n))} = 0,$$

for all $f \in C_b(\mathbb{R}^N)$ and all fixed $\varepsilon, T, n > 0$.

Next, we recall the following result. Even if it is a classical result, to the authors' knowledge it seems not to be explicitly mentioned in the most used classical books; then we give a proof.

Theorem 2.3. *Let $R > 0$ and let assumption H1 be satisfied. Moreover, assume that the coefficients of \mathcal{A} belong to $C^{k+\alpha}(\overline{B}(R))$ for some $\alpha \in (0, 1)$ and $k = 1, 2, 3$. Then for any $f \in C_0^j(B(R))$ ($j = 0, \dots, k$) the function $(t, x) \mapsto t^{(k-j)/2}(D^j T_R(t)f)(x)$ is continuous in $[0, +\infty) \times \overline{B}(R)$.*

Proof. The proof follows from a density argument. We just sketch it in the case when $j = k = 3$. It is well known that for any $f \in C_0^\infty(B(R))$, the functions $(t, x) \mapsto (D^j T_R(t)f)(x)$ ($j = 0, \dots, 3$) are continuous in $[0, +\infty) \times \overline{B}(R)$ (see e.g. [10, Theorem 5.2]). Using a method similar to the one that we will use in the proof of Theorem 3.3, we can easily show that, for any $T > 0$, there exists a positive constant $C = C(R, T)$, such that

$$(2.12) \quad \|T_R(t)f\|_{C^3(\overline{B}(R))} \leq C\|f\|_{C^3(\overline{B}(R))}, \quad \forall t \in [0, T].$$

Now, with any $f \in C_0^3(B(R))$ we associate a sequence of smooth functions $\{f_n\}_{n \in \mathbb{N}} \subset C_0^\infty(B(R))$ converging to f in $C^3(\overline{B}(R))$. Since for any $n \in \mathbb{N}$ the functions $D^j T_R(\cdot)f_n$ ($j = 0, \dots, 3$) are continuous in $[0, +\infty) \times \overline{B}(R)$, taking (2.12) into account, we easily get the assertion. \square

We conclude this section with the following version of the maximum principle which will be used throughout all this paper. For the proof, we refer the reader to [12, Proposition 2.1].

Lemma 2.4. *Suppose that assumptions H1 and H2 hold true and let $u : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a bounded classical solution of the problem*

$$\begin{cases} D_t u(t, x) = \mathcal{A}u(t, x) + g(t, x), & t \in (0, T), \quad x \in \mathbb{R}^N, \\ u(0, x) = f(x), & x \in \mathbb{R}^N, \end{cases}$$

where $f \in C_b(\mathbb{R}^N)$ and $g \in C((0, T) \times \mathbb{R}^N)$. If $g \leq 0$, then $u \leq \sup f^+$, where $f^+(x) = \max\{f(x), 0\}$. Similarly, if $g \geq 0$, then $u \geq \inf f^-$, where $f^-(x) = \min\{f(x), 0\}$.

3. UNIFORM ESTIMATES

In this section, we show that, under suitable assumptions on the coefficients q_{ij} and b_j ($i, j = 1, \dots, N$), for any $\omega > 0$, there exist positive constants $C_{k,l} = C_{k,l}(\omega)$ such that

$$(3.1) \quad \|T(t)f\|_{C_b^l(\mathbb{R}^N)} \leq C_{k,l} t^{-(l-k)/2} e^{\omega t} \|f\|_{C_b^k(\mathbb{R}^N)}, \quad \forall f \in C_b^k(\mathbb{R}^N),$$

for any $k, l \in \mathbb{N}$, $0 \leq k \leq l \leq 3$. Once (3.1) is established, one can easily prove, by interpolation, that there exist positive constants $C_{k+\alpha, m+\theta} = C_{k+\alpha, m+\theta}(\omega)$ such that

$$(3.2) \quad \|T(t)f\|_{C_b^{m+\theta}(\mathbb{R}^N)} \leq C_{k+\alpha, m+\theta} t^{-(m+\theta-k-\alpha)/2} e^{\omega t} \|f\|_{C_b^{k+\alpha}(\mathbb{R}^N)}, \quad \forall f \in C_b^{k+\alpha}(\mathbb{R}^N),$$

for any $m, k = 0, 1, 2$, $\alpha, \theta \in [0, 1]$, $k + \alpha \leq m + \theta$.

Although we limit ourselves to the case when $l \leq 3$, our techniques work as well for $l > 3$ under suitable additional assumptions on the coefficients.

We will consider the following assumptions:

H3): there exists a constant $C > 0$ such that

$$(3.3) \quad \left| \sum_{j=1}^N q_{ij}(x)x_j \right| \leq C(1 + |x|^2)\nu(x), \quad \sum_{j=1}^N q_{jj}(x) \leq C(1 + |x|^2)\nu(x),$$

$$(3.4) \quad \sum_{j=1}^N b_j(x)x_j \leq C(1 + |x|^2)\nu(x),$$

for any $x \in \mathbb{R}^N$, $i = 1, \dots, N$;

H4-1): $q_{ij}, b_j \in C_{loc}^{1+\delta}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$ and there exist a constant $C > 0$ and a function $d : \mathbb{R}^N \rightarrow \mathbb{R}$ with $L_1 := \sup_{x \in \mathbb{R}^N} \{d(x)/\nu(x)\} < +\infty$, such that $|D_k q_{ij}(x)| \leq C\nu(x)$ for any $i, j, k = 1, \dots, N$ and

$$\sum_{i,j=1}^N D_i b_j(x) \xi_i \xi_j \leq d(x) |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^N;$$

H4-2): $q_{ij}, b_j \in C_{loc}^{2+\delta}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$, hypothesis *H4-1* holds true and there exist a positive function $r : \mathbb{R}^N \rightarrow \mathbb{R}$ and three constants $K_1 \in \mathbb{R}$, $L_2, L_3 > 0$ such that $|D^\beta b_j(x)| \leq r(x)$ for any $j = 1, \dots, N$ and any $|\beta| = 2$, $d(x) + L_2 r(x) \leq L_3 \nu(x)$ and

$$\sum_{i,j,h,k=1}^N D_{hk} q_{ij}(x) m_{ij} m_{hk} \leq K_1 \nu(x) \sum_{h,k=1}^N m_{hk}^2,$$

for any symmetric matrix $M = (m_{hk})_{h,k=1}^N$ and any $x \in \mathbb{R}^N$;

H4-3): $q_{ij}, b_j \in C_{loc}^{3+\delta}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$, hypothesis *H4-2* holds true and there exists a constant $C > 0$ such that $|D^\beta b_j(x)| \leq r(x)$ and $|D^\beta q_{ij}(x)| \leq C\nu(x)$ for any $i, j = 1, \dots, N$, any $|\beta| = 3$ and any $x \in \mathbb{R}^N$.

Remark 3.1. We remark that in *H4-l* ($l = 2, 3$) we can take $r(x) = L_4(1 + |d(x)|)$ for some $L_4 > 0$. It is sufficient to take $L_2 < L_4^{-1}$ and

$$L_3 = (1 + L_2 L_4) \sup_{x \in \mathbb{R}^N} (d(x)/\nu(x)) + L_2 L_4 \nu_0^{-1}$$

in *H4-2*.

Remark 3.2. In some situation, condition *H3* is easily implied by *H4-1*. This is the case, for instance, when there exists a positive constant K such that

$$(3.5) \quad \int_0^1 \nu(tx) dt \leq K\nu(x), \quad \forall x \in \mathbb{R}^N.$$

We limit ourselves to showing that in such a situation, condition (3.4) follows from $H4-1$. The same argument can be used to prove that also (3.3) is a consequence of $H4-1$.

To show (3.4) we observe that taking $(x, \xi) = (tz, z)$ in $H4-1$ gives

$$(3.6) \quad \frac{d}{dt} \sum_{j=1}^N b_j(tz) z_j \leq L_1 \nu(tz) |z|^2, \quad \forall z \in \mathbb{R}^N.$$

Integrating (3.6) with respect to $t \in [0, 1]$ and taking (3.5) into account, we get

$$(3.7) \quad \begin{aligned} \sum_{j=1}^N b_j(z) z_j &\leq \sum_{j=1}^N b_j(0) z_j + L_1 |z|^2 \int_0^1 \nu(tz) dt \\ &\leq |z| |b(0)| + K L_1 |z|^2 \nu(z) \\ &\leq C' |z| (1 + |z|) \nu(z), \end{aligned}$$

for any $z \in \mathbb{R}^N$, where $|b(0)|$ denotes the euclidean norm of the vector $b(0) = (b_1(0), \dots, b_N(0))$ and $C' = \max\{|b(0)|\nu_0^{-1}, K L_1\}$.

Let us observe that (3.5) is satisfied, for instance, in the case when $\nu(x) = \overline{\nu}(|x|)$ for any $x \in \mathbb{R}^N$ and some nondecreasing function $\overline{\nu} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and in the case when $\nu(x) = c + \tilde{\nu}(x)$, c and $\tilde{\nu}$ being, respectively, a positive constant and a homogeneous function of degree $\alpha > 0$.

Theorem 3.3. *Let $l \in \{1, 2, 3\}$, and let assumptions $H1-H3$, $H4-l$ be satisfied. Then, for any $\omega > 0$ and any $k = 0, \dots, l$, there exist constants $C_{k,l} = C_{k,l}(\omega) > 0$ such that (3.1) holds true. In particular, if $k = l$ we can take $\omega = 0$ in (3.1).*

Proof. We begin the proof by considering the case when $k = 0$ and $l = 3$. Let $\varphi \in C^\infty([0, +\infty))$ be a nonincreasing function such that $\varphi(t) = 1$ for any $t \in [0, 1/2)$, and $\varphi(t) = 0$ for any $t \in [3/4, +\infty)$. With any $R > 0$ we associate the function $\eta = \eta_R : \mathbb{R}^N \rightarrow \mathbb{R}$ defined by $\eta(x) = \varphi(|x|/R)$. Of course $\eta \in C_0^\infty(\mathbb{R}^N)$, $\eta = 1$ in $B(R/2)$ and $\eta = 0$ outside the ball $B(R)$.

We fix $T > 0$ and define the function

$$\begin{aligned} v_{0,3,R}(t, x) &= |u_R(t, x)|^2 + at\eta^2 |Du_R(t, x)|^2 + a^2 t^2 \eta^4 |D^2 u_R(t, x)|^2 \\ &\quad + a^3 t^3 \eta^6 |D^3 u_R(t, x)|^2, \end{aligned}$$

for any $t \in [0, T]$ and any $x \in B(R)$, where $u_R(t, x) = (T_R(t)(\eta f))(x)$ is the solution of the Dirichlet Cauchy problem (2.5). Taking Theorem 2.3 into account, it follows that $v_{0,3,R}$ is continuous in $[0, T] \times \overline{B}(R)$, and, with some computations, one can see that $v_{0,3,R}$ solves the Cauchy problem

$$(3.8) \quad \begin{cases} D_t v_{0,3,R}(t, x) = \mathcal{A} v_{0,3,R}(t, x) + g_R(t, x), & t \in [0, T], \quad x \in B(R), \\ v_{0,3,R}(t, x) = 0, & t \in [0, T], \quad x \in \partial B(R), \\ v_{0,3,R}(0, x) = (\eta f)^2(x), & x \in \overline{B}(R), \end{cases}$$

where $g_R(t, x) = \sum_{j=1}^8 g_{j,R}(t, x)$ for any $t > 0$ and any $x \in B(R)$, with

$$(3.9) \quad \begin{aligned} g_{1,R} = & -2 \sum_{i,j=1}^N q_{ij} D_i u_R D_j u_R - 2at\eta^2 \sum_{i,j,h=1}^N q_{ij} D_{ih} u_R D_{jh} u_R \\ & - 2a^2 t^2 \eta^4 \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} u_R D_{jhk} u_R \end{aligned}$$

$$(3.10) \quad \begin{aligned} & - 2a^3 t^3 \eta^6 \sum_{i,j,h,k,l=1}^N q_{ij} D_{ihkl} u_R D_{jhkl} u_R, \\ (3.11) \quad g_{2,R} = & -2 \left(at |Du_R|^2 + 6a^2 t^2 \eta^2 |D^2 u_R|^2 + 15a^3 t^3 \eta^4 |D^3 u_R|^2 \right) \sum_{i,j=1}^N q_{ij} D_i \eta D_j \eta, \end{aligned}$$

$$(3.12) \quad \begin{aligned} g_{3,R} = & -2 \left(at\eta |Du_R|^2 + 2a^2 t^2 \eta^3 |D^2 u_R|^2 + 3a^3 t^3 \eta^5 |D^3 u_R|^2 \right) \sum_{i,j=1}^N q_{ij} D_{ij} \eta \\ & - 2 \left(at\eta |Du_R|^2 + 2a^2 t^2 \eta^3 |D^2 u_R|^2 + 3a^3 t^3 \eta^5 |D^3 u_R|^2 \right) \sum_{j=1}^N b_j D_j \eta \\ & - 8at\eta \sum_{i,j,h=1}^N q_{ij} D_j \eta D_h u_R D_{ih} u_R - 16a^2 t^2 \eta^3 \sum_{i,j,h,k=1}^N q_{ij} D_j \eta D_{hk} u_R D_{ihk} u_R \\ & - 24a^3 t^3 \eta^5 \sum_{i,j,h,k,l=1}^N q_{ij} D_j \eta D_{hkl} u_R D_{ihkl} u_R, \\ g_{4,R} = & 2at\eta^2 \sum_{j,h=1}^N D_h b_j D_j u_R D_h u_R + 4a^2 t^2 \eta^4 \sum_{j,h,k=1}^N D_h b_j D_{jk} u_R D_{hk} u_R \end{aligned}$$

$$(3.13) \quad \begin{aligned} & + 6a^3 t^3 \eta^6 \sum_{j,h,k,l=1}^N D_h b_j D_{jkl} u_R D_{hkl} u_R, \\ g_{5,R} = & 2at\eta^2 \sum_{i,j,h=1}^N D_h q_{ij} D_h u_R D_{ij} u_R + 4a^2 t^2 \eta^4 \sum_{i,j,h,k=1}^N D_h q_{ij} D_{hk} u_R D_{ijk} u_R \\ & + 6a^3 t^3 \eta^6 \sum_{i,j,h,k,l=1}^N D_h q_{ij} D_{hkl} u_R D_{ijkl} u_R, \end{aligned}$$

$$\begin{aligned}
g_{6,R} &= 2a^2t^2\eta^4 \sum_{i,j,h,k=1}^N D_{hk}q_{ij}D_{ij}u_R D_{hk}u_R \\
&\quad + 6a^3t^3\eta^6 \sum_{i,j,h,k,l=1}^N D_{hk}q_{ij}D_{ijl}u_R D_{hkl}u_R \\
&\quad + 2a^2t^2\eta^4 \sum_{j,h,k=1}^N D_{hk}b_jD_ju_R D_{hk}u_R \\
&\quad + 6a^3t^3\eta^6 \sum_{j,h,k,l=1}^N D_{hk}b_jD_{jl}u_R D_{hkl}u_R,
\end{aligned}
\tag{3.14}$$

$$\begin{aligned}
g_{7,R} &= 2a^3t^3\eta^6 \left(\sum_{i,j,h,k,l=1}^N D_{hkl}q_{ij}D_{ij}u_R D_{hkl}u_R + \sum_{j,h,k,l=1}^N D_{hkl}b_jD_ju_R D_{hkl}u_R \right), \\
g_{8,R} &= a\eta^2|Du_R|^2 + 2a^2t\eta^4|D^2u_R|^2 + 3a^3t^2\eta^6|D^3u_R|^2.
\end{aligned}
\tag{3.15}$$

Taking the ellipticity condition $H1$ into account, we easily deduce that

$$g_{1,R} \leq -2\nu|Du_R|^2 - 2at\eta^2\nu|D^2u_R|^2 - 2a^2t^2\eta^4\nu|D^3u_R|^2 - 2a^3t^3\eta^6\nu|D^4u_R|^2
\tag{3.16}$$

and that $g_{2,R} \leq 0$.

To estimate the function $g_{3,R}$ we observe that, by virtue of (3.3), and recalling that φ is nonincreasing in $[0, +\infty)$ and $\varphi' = 0$ in $[0, 1/2)$, it can be easily shown that

$$\begin{aligned}
&\left| \sum_{i,j=1}^N q_{ij}(x)D_{ij}\eta(x) \right| \\
&\leq |\varphi'(|x|/R)| \left(\frac{1}{|x|R} \sum_{j=1}^N q_{jj}(x) + \frac{1}{|x|^3R} \sum_{i,j=1}^N q_{ij}(x)x_ix_j \right) \\
&\quad + |\varphi''(|x|/R)| \frac{1}{|x|^2R^2} \sum_{i,j=1}^N q_{ij}(x)x_ix_j \\
&\leq C\nu(x) \sup_{|x| \leq R} \left[|\varphi'(|x|/R)| \left(\frac{1+|x|^2}{|x|R} + \frac{1+|x|^2}{|x|^3R} \sum_{i=1}^N |x_i| \right) \right. \\
&\quad \left. + C|\varphi''(|x|/R)| \frac{1+|x|^2}{|x|^2R^2} \sum_{i=1}^N |x_i| \right] \\
&\leq 4C(1+2\sqrt{N})\|\varphi'\|_\infty\nu(x) + 4C\sqrt{N}\|\varphi''\|_\infty\nu(x) \\
&=: C'\nu(x),
\end{aligned}
\tag{3.17}$$

for any $x \in B(R)$ and any $R \geq 1$. Similarly,

$$(3.19) \quad \left| \sum_{j=1}^N q_{ij}(x) D_j \eta(x) \right| \leq C\nu(x) \sup_{|x| \leq R} \left(|\varphi'(|x|/R)| \frac{1+|x|^2}{|x|R} \right) \leq 4C\|\varphi'\|_{\infty} \nu(x) =: C''\nu(x).$$

Using (3.4) and recalling that φ is nonincreasing in $[0, +\infty)$, we can easily show that

$$(3.20) \quad \begin{aligned} -\sum_{j=1}^N b_j(x) D_j \eta(x) &= -\varphi'(|x|/R) \frac{1}{|x|R} \sum_{j=1}^N b_j(x) x_j \\ &\leq C\nu(x) \sup_{|x| \leq R} \left(-\varphi'(|x|/R) \frac{1+|x|^2}{|x|R} \right) \\ &\leq C''\nu(x), \end{aligned}$$

for any $x \in B(R)$.

Taking (3.18), (3.19) and (3.20) into account and recalling that for any $a, b, \varepsilon > 0$ it holds that $ab \leq (4\varepsilon)^{-1}a^2 + \varepsilon b^2$, we can now easily show that

$$(3.21) \quad \begin{aligned} g_{3,R} &\leq 2aC't\nu\eta|Du_R|^2 + 4a^2C't^2\nu\eta^3|D^2u_R|^2 + 6a^3C't^3\nu\eta^5|D^3u_R|^2 \\ &\quad + 2aC''t\nu\eta|Du_R|^2 + 4a^2C''t^2\nu\eta^3|D^2u_R|^2 + 6a^3C''t^3\nu\eta^5|D^3u_R|^2 \\ &\quad + 8aC''t\nu \left(\frac{N}{4\varepsilon}|Du_R|^2 + \varepsilon\eta^2|D^2u_R|^2 \right) \\ &\quad + 16a^2C''t^2\nu \left(\frac{N}{4\varepsilon}\eta^2|D^2u_R|^2 + \varepsilon\eta^4|D^3u_R|^2 \right) \\ &\quad + 24a^3C''t^3\nu \left(\frac{N}{4\varepsilon}\eta^4|D^3u_R|^2 + \varepsilon\eta^6|D^4u_R|^2 \right) \\ &\leq 2a \left(C' + C'' + C''\frac{N}{\varepsilon} \right) t\nu|Du_R|^2 \\ &\quad + 4a \left(aC't + aC''t + 2C''\varepsilon + aC''t\frac{N}{\varepsilon} \right) t\nu\eta^2|D^2u_R|^2 \\ &\quad + 2a^2 \left(3aC't + 3aC''t + 8C''\varepsilon + 3aC''t\frac{N}{\varepsilon} \right) t^2\nu\eta^4|D^3u_R|^2 \\ &\quad + 24a^3C''t^3\varepsilon\nu\eta^6|D^4u_R|^2. \end{aligned}$$

Taking advantage of assumption $H4-1$, we deduce that

$$(3.22) \quad g_{4,R} \leq 2atd\eta^2|Du_R|^2 + 4a^2t^2d\eta^4|D^2u_R|^2 + 6a^3t^3d\eta^6|D^3u_R|^2.$$

The terms $g_{5,R}$, $g_{6,R}$ and $g_{7,R}$ can be estimated in a similar way by taking assumptions H_4 -2 and H_4 -3 into account, and they yield

$$(3.23) \quad \begin{aligned} g_{5,R} \leq & atC \frac{N^2}{2\varepsilon} \nu |Du_R|^2 + aCN \left(2\varepsilon + at \frac{N}{\varepsilon} \right) t\nu\eta^2 |D^2u_R|^2 \\ & + a^2CN \left(4\varepsilon + 3at \frac{N}{2\varepsilon} \right) t^2\nu\eta^4 |D^3u_R|^2 + 6a^3t^3\varepsilon CN\nu\eta^4 |D^4u_R|^2, \end{aligned}$$

$$(3.24) \quad \begin{aligned} g_{6,R} \leq & a^2t^2\eta^2r \frac{N^2}{2\varepsilon} |Du_R|^2 + a^2t^2\eta^4 \left[2K_1\nu + N \left(2\varepsilon + 3at \frac{N}{2\varepsilon} \right) r \right] |D^2u_R|^2 \\ & + 6a^3t^3\eta^6 (K_1\nu + \varepsilon Nr) |D^3u_R|^2, \end{aligned}$$

$$(3.25) \quad \begin{aligned} g_{7,R} \leq & a^3t^3\eta^2r \frac{N^3}{2\varepsilon} |Du_R|^2 + a^3t^3\eta^4 C\nu \frac{N^3}{2\varepsilon} |D^2u_R|^2 \\ & + 2a^3t^3\eta^6 \varepsilon N (C\nu N + r) |D^3u_R|^2. \end{aligned}$$

From (3.17), (3.21)-(3.25) we immediately deduce that for any $t \in [0, T]$ we have

$$(3.26) \quad \begin{aligned} g_R \leq & \left\{ -\nu_0 + a + \nu \left[-1 + 2at \left(C' + C'' + C'' \frac{N}{\varepsilon} \right) + atC \frac{N^2}{2\varepsilon} + a^2t^2C \frac{N^2}{2\varepsilon} \right] \right. \\ & \left. + at \left[2d + at(1 + atN) \frac{N^2}{2\varepsilon} r \right] \right\} |Du_R|^2 \\ & + a \left\{ -\nu_0 + 2a + \nu \left[-1 + 4 \left(aC't + aC''t + 2C''\varepsilon + aC''t \frac{N}{\varepsilon} \right) \right. \right. \\ & \left. \left. + CN \left(2\varepsilon + at \frac{N}{\varepsilon} \right) + 2K_1^+ at + a^2t^2C \frac{N^3}{2\varepsilon} \right] \right. \\ & \left. + at \left[4d + \left(2\varepsilon N + 3at \frac{N^2}{2\varepsilon} \right) r \right] \right\} t\eta^2 |D^2u_R|^2 \\ & + a^2 \left\{ -\nu_0 + 3a + \nu \left[-1 + 2 \left(3aC't + 3aC''t + 8C''\varepsilon + 3aC''t \frac{N}{\varepsilon} \right) \right. \right. \\ & \left. \left. + CN \left(4\varepsilon + 3at \frac{N}{2\varepsilon} \right) + 6atK_1^+ + 2at\varepsilon CN^2 \right] \right. \\ & \left. + 2at(3d + 4\varepsilon Nr) \right\} t^2\eta^4 |D^3u_R|^2 \\ & + 2a^3 (-1 + 12C''\varepsilon + 3\varepsilon CN) t^3\nu\eta^6 |D^4u_R|^2. \end{aligned}$$

We now choose (a, ε) , sufficiently small, satisfying the following set of inequalities:

$$\left\{ \begin{array}{l} 3a - \nu_0 \leq 0, \quad 4\varepsilon N \leq 3L_2, \\ aT(1 + aTN)\frac{N^2}{2\varepsilon} \leq 2L_2, \quad 2\varepsilon N + 3aT\frac{N^2}{2\varepsilon} \leq 4L_2, \\ -1 + aT \left[4 \left(C' + C'' + C''\frac{N}{\varepsilon} + L_3 \right) + C\frac{N^2}{\varepsilon} + 2K_1^+ \right] \\ \quad + 2\varepsilon(CN + 4C'') + a^2T^2C\frac{N^3}{2\varepsilon} \leq 0, \\ -1 + aT \left[6 \left(C' + C'' + C''\frac{N}{\varepsilon} + K_1^+ + L_3 \right) + 3C\frac{N^2}{2\varepsilon} + 2\varepsilon CN^2 \right] \\ \quad + 4\varepsilon(CN + 4C'') \leq 0. \end{array} \right.$$

With such a choice of (a, ε) we get $g_R(t, x) \leq 0$ for any $t \in [0, T]$ and any $x \in B(R)$. From the maximum principle we now deduce that

$$|v_{0,3,R}(t, x)| \leq \|\eta f\|_\infty^2 \leq \|f\|_\infty^2, \quad \forall t \in [0, T], x \in B(R).$$

By (2.11), taking the limit as R tends to $+\infty$, we deduce that (3.1) holds for $t \in [0, T]$, for some constant $C_{0,3} = C_{0,3}(T) > 0$ and with $\omega = 0$. Using the semigroup property we can then extend the estimate to all the positive t . Indeed, for any $\omega > 0$ we can choose $T = T(\omega)$ such that $e^{\omega t}t^{-3/2} \geq 1$ for all $t > T(\omega)$, and then if $t > T$ we get

$$\begin{aligned} \|T(t)f\|_{C_b^3(\mathbb{R}^N)} &= \|T(T)T(t-T)f\|_{C_b^3(\mathbb{R}^N)} \leq \frac{C_{0,3}}{T^{3/2}} \|T(t-T)f\|_\infty \\ &\leq \frac{C_{0,3}}{T^{3/2}} \|f\|_\infty \leq \frac{C_{0,3}}{T^{3/2}} \frac{e^{\omega t}}{t^{3/2}} \|f\|_\infty, \end{aligned}$$

and therefore (3.1) follows with $C_{0,3}(\omega) = \max\{C_{0,3}(T), C_{0,3}(T)T^{-3/2}\}$.

In the other cases the proof is very similar. It suffices to apply the arguments above to the function

$$v_{k,l,R}(t, x) = \sum_{j=0}^l a^j t^{(j-k)^+} (\eta(x))^{2j} |D^j u_R(t, x)|^2, \quad \forall t > 0, x \in B(R), k \leq l \leq 3.$$

Let us just show that, if $k = l$, we can take $\omega = 0$ in (3.1). We only consider the case when $l = 3$. A straightforward computation shows that $v_{3,3,R}$ is a classical solution to the Dirichlet Cauchy problem (3.8) with $v_{3,3,R}(0, \cdot) = \sum_{j=0}^3 |D^j(\eta f)|^2$ and g_R being replaced by $\tilde{g}_R = \sum_{j=1}^7 \tilde{g}_{j,R}$, where $\tilde{g}_{j,R}$ ($j = 1, \dots, 7$) are defined by the right-hand sides of (3.9)-(3.15) after replacing each t , therein explicitly appearing, with $t = 1$. Arguing as above we can easily show that \tilde{g}_R can be estimated for any $t > 0$ by the last side of (3.26), where we set $T = 1$ and replace the terms $-\nu_0 + a$, $-\nu_0 + 2a$ and $-\nu_0 + 3a$ simply by $-\nu$. It is now clear that we can take (a, ε) such that $g_R(t, x) \leq 0$ for any $t > 0$ and any $x \in \mathbb{R}^N$, and consequently

$$|v_{3,3,R}(t, x)| \leq \|\eta f\|_\infty^2 + \|D(\eta f)\|_\infty^2 + \|D^2(\eta f)\|_\infty^2 + \|D^3(\eta f)\|_\infty^2,$$

for any $t \geq 0$ and any $x \in \mathbb{R}^N$, which yields (3.1) taking the limit as R tends to $+\infty$. \square

Now, by interpolation, we can prove (3.2).

Theorem 3.4. *Let $l \in \{1, 2, 3\}$ and let assumptions $H1$ - $H3$, $H4$ - l be satisfied. Then, for any $\omega > 0$, any $k, m = 0, \dots, l-1$ and any $\alpha, \theta \in [0, 1]$ such that $k + \alpha \leq m + \theta$, there exist constants $C_{k+\alpha, m+\theta} = C_{k+\alpha, m+\theta}(\omega) > 0$ such that (3.2) holds true. In particular, if $k + \alpha = m + \theta$ we can take $\omega = 0$ in (3.2).*

Proof. The proof follows easily from an interpolation argument. We limit ourselves to sketching it in a particular case, since the same techniques can also be applied to all the other cases. So, let us assume that $l = 3$, $k = m = 2$ and $0 < \alpha \leq \theta < 1$, and fix $\omega > 0$. From (3.1) with $(k, l) = (2, 2)$ and $(k, l) = (2, 3)$, we deduce that

$$(3.27) \quad \|T(t)\|_{L(C_b^2(\mathbb{R}^N), C_b^2(\mathbb{R}^N))} \leq C_{2,2}, \quad \|T(t)\|_{L(C_b^2(\mathbb{R}^N), C_b^3(\mathbb{R}^N))} \leq C_{2,3} t^{-\frac{1}{2}} e^{\omega t},$$

for any $t > 0$. Recalling that $(C_b^2(\mathbb{R}^N); C_b^3(\mathbb{R}^N))_{\beta, \infty} = C_b^{2+\beta}(\mathbb{R}^N)$, for any $\beta \in (0, 1)$ (see [11, Theorem 1.2.17]), and applying a well-known result in interpolation (see [11, Proposition 1.2.6]), we easily see from (3.27) that $T(t)$ is bounded from $C_b^2(\mathbb{R}^N)$ into $C_b^{2+\beta}(\mathbb{R}^N)$ for any $\beta \in (0, 1)$ and

$$(3.28) \quad \|T(t)\|_{L(C_b^2(\mathbb{R}^N), C_b^{2+\beta}(\mathbb{R}^N))} \leq C_{2,2+\beta} t^{-\beta/2} e^{\beta \omega t}, \quad \forall t > 0,$$

where $C_{2,2+\beta} = C_{2,2}^{1-\beta} C_{2,3}^\beta$.

Applying the same argument to (3.1), with $(k, l) = (3, 3)$ and to (3.28), we deduce that $T(t)$ is bounded from $C_b^{2+\alpha}(\mathbb{R}^N)$ into $C_b^{2+\beta+(1-\beta)\alpha}(\mathbb{R}^N)$ and

$$\|T(t)\|_{L(C_b^{2+\alpha}(\mathbb{R}^N), C_b^{2+\beta+(1-\beta)\alpha}(\mathbb{R}^N))} \leq C_{2+\alpha, 2+\beta+(1-\beta)\alpha} t^{-(1-\alpha)\beta/2} e^{(1-\alpha)\beta \omega t},$$

for any $t > 0$, where $C_{2+\alpha, 2+\beta+(1-\beta)\alpha} = C_{2,2+\beta}^{1-\alpha} C_{3,3}^\alpha$. Now the assertion follows, taking $\beta = (\theta - \alpha)/(1 - \alpha)$. \square

In some cases, we can extend Theorems 3.3 and 3.4 to the case when in $H4$ - j , the j -th-order derivatives of the coefficients are merely continuous in \mathbb{R}^N . As the following theorem shows, this is the case when condition (3.5) is satisfied and there exist $m, M > 0$ such that

$$(3.29) \quad \int_0^1 d(tx + y) dt \leq M \nu(x + y), \quad \forall x \in \mathbb{R}^N, \quad y \in B(m).$$

As is immediately seen, the previous condition is satisfied, for instance, when d is bounded from above.

Theorem 3.5. *Suppose that assumptions $H1$ - $H3$, $H4$ - j (with the j -th-order derivatives of the coefficients merely continuous in \mathbb{R}^N) and conditions (3.5) and (3.29) are satisfied. Then, (3.1) and (3.2) hold true for any $k, l \in \mathbb{N}$, $k \leq l \leq j$.*

Proof. The proof follows from a density argument. Of course, we can limit ourselves to dealing with (3.1), since, as Theorem 3.4 shows, (3.2) follows easily from (3.1).

For any $\varepsilon > 0$, let $\varphi^\varepsilon(x) = \varepsilon^{-N} \varphi(x/\varepsilon)$, where $\varphi \in C_0^\infty(\mathbb{R}^N)$ is any nonnegative even function compactly supported in $B(1)$ with integral 1. We denote by f^ε the convolution between f and φ^ε .

Let \mathcal{A}^ε be defined as \mathcal{A} with q_{ij} and b_j being replaced, respectively, by q_{ij}^ε and b_j^ε . As it is immediately seen, q_{ij}^ε and b_j^ε ($i, j = 1, \dots, N$) satisfy assumptions $H1$, $H4$ - j , with $\nu(x)$, $d(x)$, $r(x)$ being replaced by $\nu^\varepsilon(x)$, $d^\varepsilon(x)$ and $r^\varepsilon(x)$, $C^\varepsilon = C$, $L_i^\varepsilon = L_i$ for $i = 1, 2, 3$, $K_1^\varepsilon = K_1$ (if $j > 1$) and $\nu_0^\varepsilon \geq \nu_0$.

Let us check that q_{ij}^ε and b_j^ε ($i, j = 1, \dots, N$) satisfy condition $H3$ for some positive constant *independent* of ε . For this purpose, we observe that (3.5) implies that

$$|q_{ij}(x)| \leq |q_{ij}(0)| + CK\sqrt{N}|x|\nu(x), \quad \forall x \in \mathbb{R}^N, \quad i, j = 1, \dots, N,$$

which yields (3.3).

A straightforward computation now shows that

$$|q_{ij}^\varepsilon(x)| \leq |q_{ij}(0)| + CK\sqrt{N}(|x| + \varepsilon)\nu^\varepsilon(x), \quad \forall x \in \mathbb{R}^N, \quad i, j = 1, \dots, N,$$

so that the q_{ij}^ε 's satisfy condition (3.3) with a constant *independent* of $\varepsilon \in (0, 1]$.

Similarly, we can show that the b_j^ε 's satisfy (3.4) with a positive constant *independent* of $\varepsilon \leq m$. Indeed, combining (3.7) and (3.29) we deduce that

$$\begin{aligned} \int_0^1 d^\varepsilon(tx)dt &= \int_{B(\varepsilon)} dy \int_0^1 d(tx - y)\varphi^\varepsilon(y)dt \\ &\leq M \int_{B(\varepsilon)} dy \int_0^1 \nu(x - y)\varphi^\varepsilon(y)dt \\ &= M\nu^\varepsilon(x), \end{aligned}$$

for any $x \in \mathbb{R}^N$ and $\varepsilon \leq m$. Now, arguing as in the proof of (3.7) we get

$$\sum_{j=1}^N b_j^\varepsilon(x)x_j \leq \sum_{j=1}^N b_j^\varepsilon(0)x_j + |x|^2 \int_0^1 d^\varepsilon(tx)dt \leq \sup_{y \in B(\varepsilon)} |b(y)||x| + M|x|^2\nu^\varepsilon(x),$$

for any $x \in \mathbb{R}^N$, which yields (3.4) with a constant, *independent* of $\varepsilon \leq m$.

Now, applying the same arguments as in the proof of Theorem 3.3 we can show that for any $\omega > 0$ and any $k, l \in \mathbb{N}$, $0 \leq k \leq l \leq j$, there exists a positive constant $C_{k,l} = C_{k,l}(\omega)$, *independent* of ε, R , such that

$$\|T_R^\varepsilon(t)f\|_{C^l(\overline{B}(R))} \leq C_{k,l} \frac{e^{\omega t}}{t^{(l-k)/2}} \|f\|_{C^k(\overline{B}(R))}, \quad \forall t > 0, \quad f \in C_b^k(\overline{B}(R)),$$

where T_R^ε is defined as T_R with \mathcal{A} being replaced by \mathcal{A}^ε . As R tends to $+\infty$, $T_R^\varepsilon(t)f$ tends to a solution $u^\varepsilon =: T^\varepsilon(t)f$ to the Cauchy problem (1.1) (with \mathcal{A} being replaced by \mathcal{A}^ε) which satisfies

$$(3.30) \quad \|T^\varepsilon(t)f\|_{C_b^l(\mathbb{R}^N)} \leq C_{k,l} \frac{e^{\omega t}}{t^{(l-k)/2}} \|f\|_{C_b^k(\mathbb{R}^N)}, \quad \forall t > 0$$

(see Section 2). Theorem 2.2 and (3.30) easily imply that there exists an infinitesimal sequence $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $T^{\varepsilon_n}(t)f$ and its space derivatives up to the $(j-1)$ -th-order converge in $C_{loc}^{1+\delta/2, 2+\delta}((0, +\infty) \times \mathbb{R}^N)$ to a function $u =: S(t)f$ satisfying

$$(3.31) \quad \|S(t)f\|_{C_b^l(\mathbb{R}^N)} \leq C_{k,l} \frac{e^{\omega t}}{t^{(l-k)/2}} \|f\|_{C_b^k(\mathbb{R}^N)}, \quad \forall t > 0.$$

Since the coefficients q_{ij} and b_j ($i, j = 1, \dots, N$) are locally lipschitz continuous, $q_{ij}^{\varepsilon_n}$ and $b_j^{\varepsilon_n}$ converge locally uniformly in \mathbb{R}^N as n tends to $+\infty$, respectively to q_{ij} and b_j , so that $S(t)f$ satisfies the differential equation in (1.1). Moreover, for any $f \in C_b^2(\mathbb{R}^N)$, $S(t)f$ converges to f as t tends to 0 locally uniformly in x . This can be seen by a localization argument. For this purpose, we fix $k \in \mathbb{N}$ and let $\eta = \eta_k$ be as in the proof of Theorem 3.3. Moreover, for any $n \in \mathbb{N}$ we

set $v_n(t, x) = \eta(x)(T^{\varepsilon_n}(t)f)(x)$ and observe that v_n is a solution to the Cauchy problem

$$\begin{cases} D_t v_n(t, x) = \mathcal{A}^{\varepsilon_n} v_n(t, x) + \psi_n(t, x), & (t, x) \in [0, T] \times B(k), \\ v_n(t, x) = 0, & x \in \partial B(k), \\ v_n(0, x) = \eta(x)f(x), & x \in \overline{B}(k), \end{cases}$$

where

$$\psi_n(t, \cdot) = -2T^{\varepsilon_n}(t)f \cdot \mathcal{A}^{\varepsilon_n}(\eta) - 2 \sum_{i,j=1}^N q_{ij}^{\varepsilon_n} D_i T^{\varepsilon_n}(t)f \cdot D_j \eta.$$

Since the coefficients q_{ij}^{ε} and b_j^{ε} ($i, j = 1, \dots, N$) converge locally uniformly in \mathbb{R}^N , they are equibounded (with respect to $n \in \mathbb{N}$) in $B(k)$ and there exists a positive constant \tilde{C} such that $\|\mathcal{A}^{\varepsilon_n} g\|_{C(\overline{B}(k))} \leq \tilde{C}\|g\|_{C^2(\overline{B}(k))}$ for any $g \in C^2(\overline{B}(k))$. Therefore, from (3.30) we deduce that there exists a constant \overline{C} , independent of n , such that

$$(3.32) \quad |\psi_n(t, x)| \leq \frac{\overline{C}}{t^{1/2}} \|f\|_{\infty}, \quad \forall (t, x) \in (0, T] \times B(k), \quad n \in \mathbb{N}.$$

Estimate (3.32) implies that v_n can be written by the usual variation-of-constants formula as

$$v_n(t, \cdot) = T_{k,n}(t)(\eta f) + \int_0^t T_{k,n}(t-s)\psi_n(s, \cdot)ds, \quad \forall t \in (0, T],$$

where $T_{k,n}(t)$ is the semigroup associated with the realization $A_{k,n}$ of $\mathcal{A}^{\varepsilon_n}$ in $C(\overline{B}(k))$ with Dirichlet boundary conditions (see Section 2).

Since $\eta f \in D(A_{k,n})$ (see (2.6)) and $T_{k,n}$ is a semigroup of contractions in $C(\overline{B}(k))$ for any $n \in \mathbb{N}$, then

$$\begin{aligned} \|T_{k,n}(t)(\eta f) - \eta f\|_{C(B(k))} &= \left\| \int_0^t T_{k,n}(s)\mathcal{A}^{\varepsilon_n}(\eta f)ds \right\|_{C(\overline{B}(k))} \\ &\leq t\|\mathcal{A}^{\varepsilon_n}(\eta f)\|_{C(\overline{B}(k))} \leq \overline{C}t\|\eta f\|_{C^2(\overline{B}(k))}, \quad \forall t \in (0, T), \end{aligned}$$

which readily yields

$$(3.33) \quad \lim_{t \rightarrow 0^+} \sup_{n \in \mathbb{N}} \|T_{k,n}(t)(\eta f) - \eta f\|_{C(\overline{B}(k))} = 0.$$

From (3.32) and (3.33) we can now easily show that

$$\lim_{t \rightarrow 0^+} \sup_{n \in \mathbb{N}} \|v_n(t, \cdot) - \eta f\|_{C(\overline{B}(k))} = 0.$$

Recalling that $\eta \equiv 1$ in $B(k/2)$, we deduce that

$$\begin{aligned} |(S(t)f)(x) - f(x)| &\leq \sup_{n \in \mathbb{N}} |(T^{\varepsilon_n}(t)f)(x) - \eta(x)f(x)| \\ &\quad + \limsup_{n \rightarrow +\infty} |(T^{\varepsilon_n}(t)f)(x) - (S(t)f)(x)| \\ (3.34) \quad &\leq \sup_{n \in \mathbb{N}} \|v_k(t, \cdot) - \eta f\|_{C(\overline{B}(k))}, \end{aligned}$$

for any $(t, x) \in (0, T) \times B(k/2)$. Taking the limit as t tends to 0 in both the first and the last side of (3.34) gives

$$\lim_{t \rightarrow 0^+} \sup_{x \in B(k/2)} |u(t, x) - f(x)| = 0.$$

From the arbitrariness of $k \in \mathbb{N}$, we deduce that $S(t)f$ converges locally uniformly (with respect to x) to f as t tends to 0. Hence $S(t)f$ is a classical solution to problem (1.1) and the maximum principle in Lemma 2.4 implies that $T(t)f = S(t)f$. Since $C_b^2(\mathbb{R}^N)$ is dense in $C_b^1(\mathbb{R}^N)$ (endowed with the sup-norm) we can extend the previous equality to all the $f \in C_b^1(\mathbb{R}^N)$, obtaining (3.1) in the case when $k \geq 1$.

Finally, with any $f \in C_b(\mathbb{R}^N)$ we write $T(t) = T(t/2)T(t/2)f$ and observe that since $T(t/2)f \in C_b^1(\mathbb{R}^N)$, then $T(t)f = S(t/2)T(t/2)f$. Applying (3.31) with (t, f) being replaced by $(t/2, T(t/2)f)$, we easily get (3.1) also in the case when $k = 0$. \square

Estimates (3.1) and (3.2) can be used to prove optimal Schauder estimates for the elliptic equation $\lambda u - \mathcal{A}u = f$.

Theorem 3.6. *Let $l \in \{1, 2\}$ and let assumptions H1-H3, H4-l be satisfied. Then, the maximal domain of the operator \mathcal{A} in $C_b(\mathbb{R}^N)$, i.e. the set*

$$(3.35) \quad D_{\max}(\mathcal{A}) = \left\{ u \in C_b(\mathbb{R}^N) \cap \bigcap_{1 < p < +\infty} W_{loc}^{2,p}(\mathbb{R}^N) : \mathcal{A}u \in C_b(\mathbb{R}^N) \right\},$$

is contained in $C_b^\theta(\mathbb{R}^N)$ for any $\theta \in (0, 1]$, if $l = 1$, and for any $\theta \in (0, 2)$, if $l = 2$. Moreover, for any $\omega > 0$ and any θ as above, there exists a positive constant $C = C(\theta, \omega)$ such that

$$(3.36) \quad \|u\|_{C_b^\theta(\mathbb{R}^N)} \leq C \|u\|_\infty^{1-\frac{\theta}{2}} \|(\omega - \mathcal{A})u\|_\infty^{\frac{\theta}{2}}, \quad \forall u \in D_{\max}(\mathcal{A}).$$

Finally, if assumption H4-3 is also satisfied, then for any $f \in C_b^\alpha(\mathbb{R}^N)$ ($\alpha \in (0, 1)$) and any $\lambda > 0$, there exist a unique solution $u \in C_b^{2+\alpha}(\mathbb{R}^N)$ of the elliptic equation $\lambda u - \mathcal{A}u = f$ and a positive constant $C = C(\alpha, \lambda)$ such that

$$\|u\|_{C_b^{2+\alpha}(\mathbb{R}^N)} \leq C \|f\|_{C_b^\alpha(\mathbb{R}^N)}.$$

Proof. The proof of (3.36) can be obtained from (3.1) arguing as in [12, Theorems 1 and 4.1]. \square

Taking advantage of the quoted estimates and Theorem 3.6, one can now prove sharp estimates for the solution to the Cauchy problem

$$(3.37) \quad \begin{cases} D_t u(t, x) = \mathcal{A}u(t, x) + g(t, x), & t > 0, \quad x \in \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

whose “mild” solution is given by the usual variation-of-constants formula

$$(3.38) \quad u(t, x) = (T(t)u_0)(x) + \int_0^t (T(t-s)g(s, \cdot))(x)ds, \quad t > 0, \quad x \in \mathbb{R}^N$$

(see e.g. [17, Theorem 3.5]). Arguing as in [12, Theorem 2], one can prove the following theorem.

Theorem 3.7. *Suppose that assumptions H1-H3, H4-3 are satisfied. Let $T > 0$, $\theta \in (0, 1)$ and let $g : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a bounded and continuous function such that $g(t, \cdot) \in C_b^\theta(\mathbb{R}^N)$ for every t and $\sup_{t \in [0, T]} \|g(t, \cdot)\|_{C_b^\theta(\mathbb{R}^N)} < +\infty$. Moreover, let $u_0 \in C_b^{2+\theta}(\mathbb{R}^N)$. Then, the function u in (3.38) is the unique strict solution to problem (3.37) (i.e. $D_t u, D_x u, D_{xx} u \in C_b([0, T] \times \mathbb{R}^N)$ and u solves (3.37)) and there exists a positive constant $C = C_T$ such that*

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{C_b^{2+\theta}(\mathbb{R}^N)} \leq C \left(\|u_0\|_{C_b^{2+\theta}(\mathbb{R}^N)} + \sup_{t \in [0, T]} \|g(t, \cdot)\|_{C_b^\theta(\mathbb{R}^N)} \right).$$

4. POINTWISE ESTIMATES

This section is devoted to prove pointwise estimates for the first-, second- and third-order spatial derivatives of $T(t)f$. Throughout this section we assume that conditions $H1$ - $H3$ of Section 3 hold true and we replace assumptions $H4$ with the following ones:

$H4$ -1- γ): $q_{ij}, b_j \in C_{loc}^{1+\delta}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$ and there exist $\gamma \in (0, 1)$, a constant $C > 0$ and a function $d : \mathbb{R}^N \rightarrow \mathbb{R}$ with $\sup_{x \in \mathbb{R}^N} \{d(x)/\nu(x)^\gamma\} < +\infty$, such that $|D_k q_{ij}(x)| \leq C\nu(x)^\gamma$ for any $i, j, k = 1, \dots, N$ and

$$\sum_{i,j=1}^N D_i b_j(x) \xi_i \xi_j \leq d(x) |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^N;$$

$H4$ -2- γ): $q_{ij}, b_j \in C_{loc}^{2+\delta}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$, hypothesis $H4$ -1- γ holds true and there exist $\gamma \in (0, 1)$, a positive function $r : \mathbb{R}^N \rightarrow \mathbb{R}$ with

$$\sup_{x \in \mathbb{R}^N} \{(d(x) + L_2 r(x))/\nu(x)^\gamma\} < +\infty,$$

where $L_2 := N^{3/2}/\sqrt{8}$, and two constants $C > 0$, $K_1 \in \mathbb{R}$ such that

$$|D^\beta b_j(x)| \leq r(x), \quad j = 1, \dots, N, \quad |\beta| = 2, \quad x \in \mathbb{R}^N$$

and

$$\sum_{i,j,h,k=1}^N D_{hk} q_{ij}(x) m_{ij} m_{hk} \leq K_1 \nu(x)^\gamma \sum_{h,k=1}^N m_{hk}^2,$$

for any symmetric matrix $M = (m_{hk})_{h,k=1}^N$ and any $x \in \mathbb{R}^N$;

$H4$ -3- γ): $q_{ij}, b_j \in C_{loc}^{3+\delta}(\mathbb{R}^N)$ for some $\delta \in (0, 1)$, hypothesis $H4$ -2- γ holds true with L_2 being replaced by $L_3 := 2/\sqrt{5}$ if $N = 1$ and $L_3 := \sqrt{N^3(N+1)}/3$ otherwise. Moreover, there exist $\gamma \in (0, 1)$ and a constant $C > 0$ such that $|D^\beta b_j(x)| \leq r(x)$ and $|D^\beta q_{ij}(x)| \leq C\nu(x)^\gamma$ for any $i, j = 1, \dots, N$, any $|\beta| = 3$ and any $x \in \mathbb{R}^N$.

Remark 4.1. Assumptions $H4$ - l - γ ($l = 1, 2, 3$) allow us to consider the case when the coefficients are of polynomial type.

In what follows we denote indifferently by u and $T(t)f$ the solution to problem (2.1) corresponding to the initial datum f .

We begin by proving the following lemma which will be essential to prove the first type of pointwise estimates.

Lemma 4.2. *Let $k \in \{1, 2, 3\}$ and let assumptions $H1$ - $H3$ and $H4$ - k be satisfied. Then, if $f \in C_b^k(\mathbb{R}^N)$ the function $(t, x) \mapsto (D^k T(t)f)(x)$ is continuous in $[0, +\infty) \times \mathbb{R}^N$.*

Proof. Let $f \in C_b^k(\mathbb{R}^N)$ and $u(t, x) = (T(t)f)(x)$. The regularity of u for $t > 0$ is a classical result, recalled in Theorem 2.2. Thus we have only to prove the regularity at $t = 0$. The proof is based on a localization argument.

Fix $x_0 \in \mathbb{R}^N$, let Ω be a smooth bounded neighborhood of x_0 and let $\vartheta \in C_0^\infty(\Omega)$ be such that $\vartheta \equiv 1$ in a smaller neighborhood $\Omega_0 \subset \Omega$ of x_0 . Set $v(t, x) = \vartheta(x)u(t, x)$ for $t > 0$ and $x \in \Omega$; then the function v satisfies the equation

$$v_t(t, x) - \mathcal{A}v(t, x) = \psi(t, x) = -u(t, x)\mathcal{A}\vartheta(x) - 2 \sum_{i,j=1}^N q_{ij}(x) D_i u(t, x) D_j \vartheta(x),$$

for all $t > 0$ and $x \in \Omega$, and the boundary condition $v(t, x) = 0$ for all $t > 0$ and $x \in \partial\Omega$. Moreover, it is readily seen that there exists a constant $C > 0$ such that

$$\|\psi(t, \cdot)\|_\infty \leq C\|u(t, \cdot)\|_{C^1(\overline{\Omega})} \leq C \frac{C_{0,1}e^{\omega T}}{\sqrt{t}} \|f\|_\infty, \quad 0 < t \leq T.$$

Here, we have used the estimate (3.1) with $(k, l) = (0, 1)$. In particular, the function $t \mapsto \|\psi(t, \cdot)\|_\infty$ belongs to $L^1(0, T)$ for all $T > 0$, and therefore we can write

$$v(t, x) = (T_\Omega(t)(\vartheta f))(x) = \int_0^t (T_\Omega(t-s)\psi(s, \cdot))(x) ds, \quad \forall t > 0, x \in \Omega,$$

where $T_\Omega(t)$ is the semigroup associated with the Cauchy problem with Dirichlet boundary condition in Ω . Now let $k = 1, 2, 3$ and $f \in C_b^k(\mathbb{R}^N)$; using the classical gradient estimates for $T_\Omega(t)$ and the estimate (3.1), we get

$$\begin{aligned} \|D^k T_\Omega(t-s)\psi(s, \cdot)\|_\infty &\leq \frac{C_T}{\sqrt{t-s}} \|\psi(s, \cdot)\|_{C^{k-1}(\overline{\Omega})} \\ &\leq C \frac{C_T}{\sqrt{t-s}} \|u(s, \cdot)\|_{C^k(\overline{\Omega})} \\ &\leq C_{k,k} C \frac{C_T}{\sqrt{t-s}} \|f\|_{C^k(\mathbb{R}^N)}, \end{aligned}$$

for any $0 < s < t \leq T$, where $C, C_T > 0$ are constants. This means that the function $t \mapsto \|D^k T_\Omega(t-s)\psi(s, \cdot)\|_\infty$ belongs to $L^1(0, T)$ for all $T > 0$, and therefore we can write

$$D^k v(t, x) - (D^k T_\Omega(t)(\vartheta f))(x) = \int_0^t (D^k T_\Omega(t-s)\psi(s, \cdot))(x) ds, \quad t > 0, x \in \Omega,$$

and

$$|D^k v(t, x) - (D^k T_\Omega(t)(\vartheta f))(x)| \leq \int_0^t C \frac{C_{T,k}}{\sqrt{t-s}} \|f\|_{C^k(\overline{\Omega})} ds, \quad 0 < t \leq T, x \in \Omega.$$

This implies that the function $(t, x) \mapsto D^k v(t, x)$ is continuous in $[0, T] \times \Omega_0$ since, by virtue of Theorem 2.3, $(D^k T_\Omega(t)(\vartheta f))(x)$ tends to $D^k(\vartheta f)(x)$ uniformly in Ω_δ , for any $\delta > 0$, as t tends to 0. Since $v \equiv u$ in a neighborhood of x_0 , it follows that $D^k u(t, x)$ is continuous at $(0, x_0)$. \square

We now define the following quantities which will be widely used in what follows. We set

$$(4.1) \quad M_1 = \sup_{x \in \mathbb{R}^N} \{d(x)/\nu(x)^\gamma\}, \quad M_k = \sup_{x \in \mathbb{R}^N} \{(d(x) + L_k r(x))/\nu(x)^\gamma\} \quad (k = 2, 3),$$

where L_k ($k = 2, 3$) is defined in assumption H_4 - k - γ .

We are now able to prove the following theorem in which we assume that $M_k \geq 0$.

Theorem 4.3. *Let $l \in \{1, 2, 3\}$ and let assumptions H_1 - H_3 , H_4 - l - γ be satisfied. Further assume that $M_l \geq 0$. Then, for any $f \in C_b^l(\mathbb{R}^N)$ and any $p \in (1, +\infty)$*

(4.2)

$$|(D^l T(t)f)(x)|^p \leq e^{\sigma_{l,p} t} \left(T(t) \left[\left(\sum_{j=0}^l |D^j f|^2 \right)^{\frac{p}{2}} \right] \right)(x), \quad \forall t > 0, x \in \mathbb{R}^N,$$

where, if $p \in (1, 2]$,

$$\sigma_{l,p} = p \sup_{x \in \mathbb{R}^N} \left\{ (1-p)\nu(x) + \frac{C_l(p)}{p-1} (\nu(x))^\gamma \right\} \vee 0,$$

$C_l(p)$ being a suitable positive constant (that can be explicitly determined) such that $\lim_{p \rightarrow 1} C_l(p) \in \mathbb{R}$, while, if $p > 2$, $\sigma_{l,p} = \frac{p}{2}\sigma_{l,2}$ ($l = 1, 2, 3$).

Proof. We begin with the case when $l = 3$ and $p \in (1, 2]$. For any $\delta > 0$ we introduce the function $w_\delta : \overline{\mathbb{R}}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$w_\delta(t, x) = (|u(t, x)|^2 + |Du(t, x)|^2 + |D^2u(t, x)|^2 + |D^3u(t, x)|^2 + \delta)^{\frac{p}{2}},$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. By Lemma 4.2, $w_\delta \in C_b([0, +\infty) \times \mathbb{R}^N) \cap C_{loc}^{1,2}((0, +\infty) \times \mathbb{R}^N)$ and a straightforward computation shows that it solves the Cauchy problem

$$(4.3) \quad \begin{cases} D_t w_\delta(t, x) = \mathcal{A}w_\delta(t, x) + g_\delta(t, x), & t > 0, x \in \mathbb{R}^N, \\ w_\delta(0, x) = (|f(x)|^2 + |Df(x)|^2 + |D^2f(x)|^2 + |D^3f(x)|^2 + \delta)^{\frac{p}{2}}, & x \in \mathbb{R}^N, \end{cases}$$

where

$$\begin{aligned} g_\delta = & p (|u|^2 + |Du|^2 + |D^2u|^2 + |D^3u|^2 + \delta)^{\frac{p}{2}-1} \\ & \times \left(- \sum_{i,j=1}^N q_{ij} D_i u D_j u - \sum_{i,j,h=1}^N q_{ij} D_{ih} u D_{jh} u - \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} u D_{jhk} u \right. \\ & - \sum_{i,j,h,k,l=1}^N q_{ij} D_{ihkl} u D_{jhkl} u + \sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u \\ & + 2 \sum_{i,j,h,k=1}^N D_h q_{ij} D_{hk} u D_{ijk} u + 3 \sum_{i,j,h,k,l=1}^N D_h q_{ij} D_{hkl} u D_{ijkl} u \\ & + \sum_{j,h=1}^N D_h b_j D_j u D_h u + 2 \sum_{j,h,k=1}^N D_h b_j D_{jk} u D_{hk} u \\ & + 3 \sum_{j,h,k,l=1}^N D_h b_j D_{jkl} u D_{hkl} u + \sum_{i,j,h,k=1}^N D_{hk} q_{ij} D_{ij} u D_{hk} u \\ & + 3 \sum_{i,j,h,k,l=1}^N D_{hk} q_{ij} D_{ijl} u D_{hkl} u + \sum_{j,h,k=1}^N D_{hk} b_j D_j u D_{hk} u \\ & + 3 \sum_{j,h,k,l=1}^N D_{hk} b_j D_{jl} u D_{hkl} u + \sum_{i,j,h,k,l=1}^N D_{hkl} q_{ij} D_{ij} u D_{hkl} u \\ & \left. + \sum_{j,h,k,l=1}^N D_{hkl} b_j D_j u D_{hkl} u \right) \end{aligned}$$

$$\begin{aligned}
& + p(2-p) (|u|^2 + |Du|^2 + |D^2u|^2 + |D^3u|^2 + \delta)^{\frac{p}{2}-2} \\
& \times \sum_{i,j=1}^N q_{ij} \left(uD_iu + \sum_{h=1}^N D_huD_{ih}u + \sum_{h,k=1}^N D_{hk}uD_{ihk}u \right. \\
& \quad \left. + \sum_{h,k,l=1}^N D_{hkl}uD_{ihkl}u \right) \\
& \times \left(uD_ju + \sum_{h=1}^N D_huD_{jh}u + \sum_{h,k=1}^N D_{hk}uD_{jhk}u \right. \\
& \quad \left. + \sum_{h,k,l=1}^N D_{hkl}uD_{jhkl}u \right).
\end{aligned}$$

Now, let $h, k \in \{0, 1, 2, 3\}$ be fixed. Applying the Cauchy-Schwarz inequality twice (first to the inner product $(\xi, \eta) \mapsto \langle Q(x)\xi, \eta \rangle$ and then to the euclidean one) we deduce that

$$\begin{aligned}
(4.4) \quad & \sum_{i,j=1}^N q_{ij} \sum_{|\alpha|=h} D^\alpha u D_i D^\alpha u \sum_{|\beta|=k} D^\beta u D_j D^\beta u \\
& = \sum_{|\alpha|=h} \sum_{|\beta|=k} D^\alpha u D^\beta u \sum_{i,j=1}^N q_{ij} D_i D^\alpha u D_j D^\beta u \\
& \leq \sum_{|\alpha|=h} |D^\alpha u| \left(\sum_{i,j=1}^N q_{ij} D_i D^\alpha u D_j D^\alpha u \right)^{\frac{1}{2}} \sum_{|\beta|=k} |D^\beta u| \left(\sum_{i,j=1}^N q_{ij} D_i D^\beta u D_j D^\beta u \right)^{\frac{1}{2}} \\
& \leq |D^h u| |D^k u| \left(\sum_{|\alpha|=h} \sum_{i,j=1}^N q_{ij} D_i D^\alpha u D_j D^\alpha u \right)^{\frac{1}{2}} \left(\sum_{|\beta|=k} \sum_{i,j=1}^N q_{ij} D_i D^\beta u D_j D^\beta u \right)^{\frac{1}{2}}.
\end{aligned}$$

This estimate can be used in order to get

$$\begin{aligned}
(4.5) \quad & \sum_{i,j=1}^N q_{ij} \left(uD_iu + \sum_{h=1}^N D_huD_{ih}u + \sum_{h,k=1}^N D_{hk}uD_{ihk}u + \sum_{h,k,l=1}^N D_{hkl}uD_{ihkl}u \right) \\
& \times \left(uD_ju + \sum_{h=1}^N D_huD_{jh}u + \sum_{h,k=1}^N D_{hk}uD_{jhk}u + \sum_{h,k,l=1}^N D_{hkl}uD_{jhkl}u \right) \\
& \leq \left[|u| \left(\sum_{i,j=1}^N q_{ij} D_i u D_j u \right)^{\frac{1}{2}} + |Du| \left(\sum_{i,j,h=1}^N q_{ij} D_{ih} u D_{jh} u \right)^{\frac{1}{2}} \right. \\
& \quad \left. + |D^2u| \left(\sum_{i,j,h,k=1}^N q_{ij} D_{ihk} u D_{jhk} u \right)^{\frac{1}{2}} \right]
\end{aligned}$$

$$\begin{aligned}
& + |D^3 u|^2 \left(\sum_{i,j,h,k,l=1}^N q_{ij} D_{ihkl} u D_{jhkl} u \right)^{\frac{1}{2}} \Big]^2 \\
& \leq (|u|^2 + |Du|^2 + |D^2 u|^2 + |D^3 u|^2) \\
& \quad \times \left(\sum_{i,j=1}^N q_{ij} D_i u D_j u + \sum_{i,j,h=1}^N q_{ij} D_{ih} u D_{jh} u \right. \\
& \quad \left. + \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} u D_{jhk} u + \sum_{i,j,h,k,l=1}^N q_{ij} D_{ihkl} u D_{jhkl} u \right).
\end{aligned}$$

Taking (4.5) into account, it is immediate to check that

$$\begin{aligned}
g_\delta \leq p \Big\{ (1-p) \Big[\sum_{i,j=1}^N q_{ij} D_i u D_j u + \sum_{i,j,h=1}^N q_{ij} D_{ih} u D_{jh} u \\
+ \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} u D_{jhk} u + \sum_{i,j,h,k,l=1}^N q_{ij} D_{ihkl} u D_{jhkl} u \Big] \\
+ \sum_{i,j,h=1}^N D_h q_{ij} D_h u D_{ij} u + 2 \sum_{i,j,h,k=1}^N D_h q_{ij} D_{hk} u D_{ijk} u \\
+ 3 \sum_{i,j,h,k,l=1}^N D_h q_{ij} D_{hkl} u D_{ijkl} u + \sum_{j,h=1}^N D_h b_j D_j u D_h u \\
+ 2 \sum_{j,h,k=1}^N D_h b_j D_{jk} u D_{hk} u + 3 \sum_{j,h,k,l=1}^N D_h b_j D_{jkl} u D_{hkl} u \\
+ \sum_{i,j,h,k=1}^N D_{hk} q_{ij} D_{ij} u D_{hk} u + 3 \sum_{i,j,h,k,l=1}^N D_{hk} q_{ij} D_{ijl} u D_{hkl} u \\
+ \sum_{j,h,k=1}^N D_{hk} b_j D_j u D_{hk} u + 3 \sum_{j,h,k,l=1}^N D_{hk} b_j D_{jl} u D_{hkl} u \\
+ \sum_{i,j,h,k,l=1}^N D_{hkl} q_{ij} D_{ij} u D_{hkl} u + \sum_{j,h,k,l=1}^N D_{hkl} b_j D_j u D_{hkl} u \Big\} \\
\times (|u|^2 + |Du|^2 + |D^2 u|^2 + |D^3 u|^2 + \delta)^{\frac{p}{2}-1}.
\end{aligned}$$

Hence, using hypotheses $H1$ and $H4$ - β - γ , we get for all $\varepsilon, \varepsilon_1 > 0$

$$\begin{aligned}
g_\delta \leq p \Big\{ \Big[(1-p)\nu + C \frac{N^2}{4\varepsilon_0} \nu^\gamma + d + r \frac{N^2}{4\varepsilon_1} (N+1) \Big] |Du|^2 \\
+ \Big[(1-p)\nu + \left(C\varepsilon_0 N + C \frac{N^2}{2\varepsilon} + C \frac{N^3}{4\varepsilon} + K_1 \right) \nu^\gamma \\
+ 2d + rN \left(\varepsilon_1 + \frac{3N}{4\varepsilon_1} \right) \Big] |D^2 u|^2
\end{aligned}$$

$$\begin{aligned}
(4.6) \quad & + \left[(1-p)\nu + \left(2C\varepsilon N + C\frac{3N^2}{4\varepsilon} + \varepsilon CN^2 + 3K_1 \right) \nu^\gamma \right. \\
& \quad \left. + 3d + 4r\varepsilon_1 N \right] |D^3 u|^2 \\
& + \left((1-p)\nu^{1-\gamma} + 3C\varepsilon N \right) \nu^\gamma |D^4 u|^2 \Big\} \\
& \times (|u|^2 + |Du|^2 + |D^2 u|^2 + |D^3 u|^2 + \delta)^{\frac{p}{2}-1}.
\end{aligned}$$

We now choose $\varepsilon = \varepsilon_0 = (p-1)\nu_0^{1-\gamma}/(3CN)$ and ε_1 to be the minimazing point of the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$g(x) = N \max\{(N^2 + N)/(4x), (4x^2 + 3N)/(8x), 4x/3\}, \quad \forall x \in \mathbb{R}^N.$$

We obtain $\varepsilon_1 = 3\sqrt{5}/10$ if $N = 1$ and $\varepsilon_1 = \sqrt{3N(N+1)}/4$ otherwise. With this choice of ε and ε_1 , we get

$$\begin{aligned}
g_\delta \leq p \Big[& \left((1-p)\nu + \tilde{C}_1(p)\nu^\gamma \right) |Du|^2 + \left((1-p)\nu + \tilde{C}_2(p)\nu^\gamma \right) |D^2 u|^2 \\
& + \left((1-p)\nu + \tilde{C}_3(p)\nu^\gamma \right) |D^3 u|^2 \Big] \\
& \times (|u|^2 + |Du|^2 + |D^2 u|^2 + |D^3 u|^2 + \delta)^{\frac{p}{2}-1},
\end{aligned}$$

where

$$\begin{aligned}
\tilde{C}_1(p) &= \frac{3C^2 N^3}{4(p-1)} \nu_0^{\gamma-1} + M_3, \\
\tilde{C}_2(p) &= \frac{1}{3}(p-1)\nu_0^{1-\gamma} + \frac{3C^2 N^3(N+2)}{4(p-1)} \nu_0^{\gamma-1} + K_1 + 2M_3, \\
\tilde{C}_3(p) &= \frac{1}{3}(N+2)(p-1)\nu_0^{1-\gamma} + \frac{9C^2 N^3}{4(p-1)} \nu_0^{\gamma-1} + 3K_1 + 3M_3,
\end{aligned}$$

with M_3 being given by (4.1). Now let $\sigma_{3,p}$ be as in the statement of the theorem with $C_3(p) = (p-1) \max\{\tilde{C}_1(p), \tilde{C}_2(p), \tilde{C}_3(p)\}$. Then, we have

$$\begin{aligned}
g_\delta(t, x) &\leq \sigma_{3,p} (|Du|^2 + |D^2 u|^2 + |D^3 u|^2) (|u|^2 + |Du|^2 + |D^2 u|^2 + |D^3 u|^2 + \delta)^{\frac{p}{2}-1} \\
&\leq \sigma_{3,p} w(t, x),
\end{aligned}$$

for any $t > 0$ and any $x \in \mathbb{R}^N$. This yields

$$D_t w_\delta(t, x) \leq \mathcal{A} w_\delta(t, x) + \sigma_{3,p} w_\delta(t, x).$$

Now set $z_\delta(t, x) = e^{-\sigma_{3,p} t} w_\delta(t, x)$ and observe that z_δ satisfies

$$(4.7) \quad \begin{cases} D_t z_\delta(t, x) \leq \mathcal{A} z_\delta(t, x), & t > 0, \quad x \in \mathbb{R}^N, \\ z_\delta(0, x) \leq (|f(x)|^2 + |Df(x)|^2 + |D^2 f(x)|^2 + |D^3 f(x)|^2 + \delta)^{\frac{p}{2}}, & x \in \mathbb{R}^N. \end{cases}$$

Then, the maximum principle in Lemma 2.4 implies that

$$z_\delta(t, x) \leq \left(T(t) (|f|^2 + |Df|^2 + |D^2f|^2 + |D^3f|^2)^{\frac{p}{2}} \right) (x), \quad t > 0, \quad x \in \mathbb{R}^N,$$

for any $\delta > 0$. Taking the limit as δ tends to 0 we get

$$(4.8) \quad \begin{aligned} & (|u(t, x)|^2 + |Du(t, x)|^2 + |D^2u(t, x)|^2 + |D^3u(t, x)|^2)^{\frac{p}{2}} \\ & \leq e^{\sigma_{3,p}t} \left(T(t) (|f|^2 + |Df|^2 + |D^2f|^2 + |D^3f|^2)^{\frac{p}{2}} \right) (x), \end{aligned}$$

for all $t > 0$ and $x \in \mathbb{R}^N$. Then, (4.2) follows.

Finally, we consider the case when $p > 2$. Since

$$\begin{aligned} |D^3T(t)f(x)|^p &= (|D^3T(t)f(x)|^2)^{\frac{p}{2}} \\ &\leq (e^{\sigma_{3,2}t} (T(t) (|f|^2 + |Df|^2 + |D^2f|^2 + |D^3f|^2)) (x))^{\frac{p}{2}}, \end{aligned}$$

we get (4.2) (with $l = 3$) observing that (2.2) and (2.3) easily imply that

$$(T(t)\psi)^{p/2} \leq T(t)(\psi^{p/2}),$$

for any $t > 0$ and any nonnegative $\psi \in C_b(\mathbb{R}^N)$.

To get (4.2) in the case when $p \in (1, 2)$ and $l = 1, 2$, it suffices to apply the previous arguments to the function

$$w_\delta(t, x) = (|(T(t)f)(x)|^2 + |(DT(t)f)(x)|^2 + \delta)^{\frac{p}{2}}, \quad t > 0, \quad x \in \mathbb{R}^N, \quad \delta > 0,$$

if $l = 1$ and to the function

$$w_\delta(t, x) = (|(T(t)f)(x)|^2 + |(DT(t)f)(x)|^2 + (|D^2T(t)f(x)|^2 + \delta)^{\frac{p}{2}}), \quad \delta > 0,$$

if $l = 2$. Arguing as above we get

$$(4.9) \quad \left(\sum_{j=0}^l |D^j u(t, x)|^2 + \delta \right)^{\frac{p}{2}} \leq e^{\sigma_{l,p}t} \left(T(t) \left(\sum_{j=0}^l |D^j f|^2 + \delta \right)^{\frac{p}{2}} \right) (x) + \delta^{\frac{p}{2}},$$

for any $l = 1, 2$, with

$$(4.10) \quad \frac{C_1(p)}{p-1} = \frac{C^2 N^3 \nu_0^{\gamma-1}}{4(p-1)} + M_1,$$

$$(4.11) \quad \frac{C_2(p)}{p-1} = \max \left\{ \frac{C^2 N^3 \nu_0^{\gamma-1}}{2(p-1)} + M_2, \frac{p-1}{2} \nu_0^{1-\gamma} + \frac{C^2 N^3 \nu_0^{\gamma-1}}{p-1} + K_1 + 2M_2 \right\},$$

where M_1 and M_2 are given by (4.1). Taking the limit as δ tends to 0 gives (4.2).

In the case when $p > 2$, (4.2) (with $l = 1, 2$) follows from the case $p = 2$, applying Jensen's inequality. This finishes the proof. \square

As the following theorem shows, in the case when $M_k < 0$ ($k = 2, 3$) the result in Theorem 4.3 can be improved.

Theorem 4.4. *Let $l = 1, 2, 3$ and let assumptions H1-H3, H_4 - l - γ be satisfied. Further, assume that $M_l < 0$ (if $l = 2, 3$) or $M_1 \leq -\frac{C^2 N^3}{4(p_0-1)} \nu_0^{\gamma-1}$ for some $p_0 \in (1, 2)$ if $l = 1$. Then*

$$(4.12) \quad \left(\sum_{j=1}^l |(D^j T(t)f)(x)|^2 \right)^{\frac{p}{2}} \leq e^{\bar{\sigma}_{k,p}t} \left(T(t) \left(\sum_{j=1}^l |D^j f|^2 \right)^{\frac{p}{2}} \right) (x), \quad \forall t > 0, \quad x \in \mathbb{R}^N,$$

for all $f \in C_b^l(\mathbb{R}^N)$ and any $p \in (1, +\infty)$ if $l = 2, 3$ and any $p \in [p_0, +\infty)$ if $l = 1$, where $\bar{\sigma}_{l,p}$ ($p \in (1, +\infty)$) are suitable constants that can be explicitly determined (see the proof) and they possibly blow up as p tends to 1.

Proof. The proof is close to that of Theorem 4.3. Hence we just sketch it. To get (4.12), in the case when $p \in (1, 2]$ and $l = 3$, one has to deal with the function

$$\bar{w}_{\delta,3}(t, x) = (|Du(t, x)|^2 + |D^2u(t, x)|^2 + |D^3u(t, x)|^2 + \delta)^{\frac{p}{2}}, \quad \forall t > 0, x \in \mathbb{R}^N.$$

A straightforward computation shows that $\bar{w}_{\delta,3}$ solves problem (4.3) with g_δ replaced by a function $\bar{g}_\delta \in C([0, +\infty) \times \mathbb{R}^N)$ which satisfies estimate (4.6), where we drop out the term $-(1-p)\nu$ in the first round brackets and the term $|u|^2$ in the last ones. We now choose $\varepsilon_0 > -CN^2/(4M_3)$, ε and ε_1 as in the proof of Theorem 4.3, and we set

$$C_1(p, \varepsilon_0) = \left(\frac{CN^2}{4\varepsilon_0} + M_3 \right) \nu_0^\gamma,$$

$$C_2(p, \varepsilon_0) = \sup_{\mathbb{R}^N} \left((1-p)\nu + \left(C\varepsilon_0 N + \frac{3C^2N^3(N+2)}{4(p-1)} \nu_0^{\gamma-1} + K_1 + 2M_3 \right) \nu^\gamma \right),$$

$$C_3(p) = \sup_{\mathbb{R}^N} \left((1-p)\nu + \tilde{C}_3(p)\nu^\gamma \right)$$

and $\bar{\sigma}_{3,p} = p \min_{\varepsilon_0 > -\frac{CN^2}{4M_3}} \{ \max \{ C_1(p, \varepsilon_0), C_2(p, \varepsilon_0), C_3(p) \} \}$. We get

$$\begin{aligned} \bar{g}_\delta &\leq \bar{\sigma}_{3,p} (|Du|^2 + |D^2u|^2 + |D^3u|^2) (|Du|^2 + |D^2u|^2 + |D^3u|^2 + \delta)^{\frac{p}{2}-1} \\ &\leq \bar{\sigma}_{3,p} \bar{w}_{\delta,3} - \bar{\sigma}_{3,p} \delta^{\frac{p}{2}}. \end{aligned}$$

If $\bar{\sigma}_{3,p} \geq 0$, the proof follows as in Theorem 4.3. In the case when $\bar{\sigma}_{3,p} < 0$, we introduce the function $\bar{z}_{\delta,3}(t, x) = e^{-\bar{\sigma}_{3,p}t} (\bar{w}_{\delta,3}(t, x) - \delta^{\frac{p}{2}})$ which satisfies (4.7). The maximum principle in Lemma 2.4 implies that

$$\bar{z}_{\delta,3}(t, x) \leq \left(T(t) (|Df|^2 + |D^2f|^2 + |D^3f|^2)^{\frac{p}{2}} \right) (x),$$

which gives (4.12) taking the limit as δ tends to 0. The case when $p > 2$ follows from the case when $p = 2$ and from Hölder's inequality, and gives $\bar{\sigma}_{3,p} = \frac{p}{2} \bar{\sigma}_{3,2}$.

Applying the same arguments respectively to the functions

$$\bar{w}_{\delta,1} = (|Du|^2 + \delta)^{\frac{p}{2}}, \quad \bar{w}_{\delta,2} = (|Du|^2 + |D^2u|^2 + \delta)^{\frac{p}{2}},$$

one finally gets (4.12) with $\bar{\sigma}_{1,p} = p(p_0 - p)\nu_0$ and

$$\bar{\sigma}_{2,p} = p \min_{\varepsilon_0 > -\frac{CN^2}{4M_2}} \{ \max \{ E_1(\varepsilon_0), E_2(\varepsilon_0) \} \},$$

where

$$E_1(\varepsilon_0) = \left(\frac{CN^2}{4\varepsilon_0} + M_2 \right) \nu_0^\gamma$$

and

$$E_2(\varepsilon_0) = \sup_{\mathbb{R}^N} \left[(1-p)\nu + \left(C\varepsilon_0 N + \frac{C^2N^3}{p-1} \nu_0^{\gamma-1} + K_1 + 2M_2 \right) \nu^\gamma \right],$$

if $p \in (1, 2]$, and $\bar{\sigma}_{l,p} = \frac{p}{2} \bar{\sigma}_{l,2}$ ($l = 1, 2$) if $p > 2$. \square

Remark 4.5. In the case when $l = 1$, estimate (4.12) has already been proved in [4] under assumption $H4$ -1- γ with $\gamma = 1/2$ for any value of the constant M_1 .

We now consider the second type of pointwise estimates we want to prove. First we prove the following lemma.

Lemma 4.6. *If $g_R, g : [0, T] \times \mathbb{R}^N$, $R > 0$, are continuous functions such that $\|g_R\|_\infty \leq M$ for all $R > 0$ and $\lim_{R \rightarrow +\infty} g_R = g$, uniformly in $[\varepsilon, T - \varepsilon] \times B(k)$ for every $\varepsilon, k > 0$, then*

$$\lim_{R \rightarrow +\infty} (T_R(t)g_R(t))(x) = (T(t)g(t))(x),$$

uniformly in $[\varepsilon, T - \varepsilon] \times B(k)$ for every $\varepsilon, k > 0$.

If $f_\varepsilon, f_0 \in C_b(\mathbb{R}^N)$, $\varepsilon > 0$, are such that $\|f_\varepsilon\|_\infty \leq M$ for all $\varepsilon > 0$ and $\lim_{\varepsilon \rightarrow 0^+} f_\varepsilon = f_0$, uniformly in $B(k)$ for all $k > 0$, then for all $t > 0$ we have $\lim_{\varepsilon \rightarrow 0^+} T(t - \varepsilon)f_\varepsilon = T(t)f_0$, uniformly in $B(k)$ for all $k > 0$.

Proof. Let g_R, g as above. We have

$$\begin{aligned} |T_R(t)g_R(t) - T(t)g(t)| &\leq |T_R(t)(g_R(t) - g(t))| + |(T(t) - T_R(t))g(t)| \\ &\leq T_R(t)(|g_R(t) - g(t)|) + (T(t) - T_R(t))|g(t)| \\ (4.13) \qquad \qquad \qquad &\leq T(t)(|g_R(t) - g(t)|) + M(T(t) - T_R(t))\mathbb{1}, \end{aligned}$$

for all $R > 0$. In deriving (4.13) we took advantage of (2.9).

Let us observe that the second term in the last part of (4.13) converges to zero uniformly in $[\varepsilon, T] \times B(k)$ for every $\varepsilon, k > 0$, by (2.7). To estimate the first term, we observe that for any $t \in [\varepsilon, T - \varepsilon]$ and any $x \in \mathbb{R}^N$ we have

$$(4.14) \qquad (T(t)(|g_R(t) - g(t)|))(x) \leq (T(t)(\sup_{t \in [\varepsilon, T - \varepsilon]} |g_R(t) - g(t)|))(x).$$

Since $\sup_{t \in [\varepsilon, T - \varepsilon]} |g_R(t) - g(t)|$ is a bounded and continuous function in \mathbb{R}^N , converging to 0, locally uniformly in \mathbb{R}^N , from (2.4) we deduce that the right-hand side of (4.14) converges to 0, locally uniformly, as R tends to $+\infty$, and the conclusion follows.

We prove the second statement. Let f_ε, f_0 be as above; we have

$$(4.15) \quad |T(t - \varepsilon)f_\varepsilon - T(t)f_0| \leq |T(t - \varepsilon)f_\varepsilon - T(t - \varepsilon)f_0| + |T(t - \varepsilon)f_0 - T(t)f_0|,$$

for all $\varepsilon > 0$. By (2.4), for any fixed $t > 0$, the first term in (4.15) converges to 0 as ε tends to 0, locally uniformly in $B(k)$ for any $k > 0$. The second term in the right-hand side of (4.15) converges to zero uniformly in $B(k)$ for any $k > 0$ as well, since the function $T(t)f_0$ is continuous in $[0, +\infty) \times \mathbb{R}^N$ and, consequently, uniformly continuous in $[0, T] \times B(k)$ for any $T, k > 0$. \square

We can now prove our estimates. For notation convenience, we set

$$\hat{\sigma}_{1,p} = \begin{cases} \sigma_{1,p}, & \text{if } M_1 \geq -\frac{C^2 N^3}{4(p_0 - 1)} \nu_0^{\gamma-1} \text{ and } p \geq p_0, \\ \bar{\sigma}_{1,p}, & \text{otherwise,} \end{cases} \qquad \hat{\sigma}_{l,p} = \begin{cases} \sigma_{l,p}, & \text{if } M_l \geq 0, \\ \bar{\sigma}_{l,p}, & \text{otherwise,} \end{cases}$$

for $l = 2, 3$, where $\sigma_{l,p}$ and $\bar{\sigma}_{l,p}$ are defined, respectively, in Theorems 4.3 and 4.4.

Theorem 4.7. *Let $l \in \{1, 2, 3\}$ and suppose that assumptions H1-H3, and H4- $l-\gamma$ hold. Then, for any $f \in C_b^{l-1}(\mathbb{R}^N)$ we have*

$$(4.16) \qquad |D^l T(t)f(x)|^p \leq \left(\frac{\hat{\sigma}_{l,p \wedge 2}}{1 - e^{-\hat{\sigma}_{l,p \wedge 2} t}} \phi_{l,p \wedge 2}(t) \right)^{\max\{1, \frac{p}{2}\}} \left(T(t) \left(\sum_{j=0}^{l-1} |D^j f|^2 \right)^{\frac{p}{2}} \right)(x),$$

for any $t > 0$ and $x \in \mathbb{R}^N$, where $\phi_{1,r}(t) = (r(r-1)\nu_0)^{-\frac{r}{2}} t^{1-\frac{r}{2}} + t$ for any $r \in (1, 2]$, and

$$\phi_{k,r}(t) = a_{r,k} \left(\frac{e^{\hat{\sigma}_{k-1,r}t} - 1}{\hat{\sigma}_{k-1,r}} \right)^{1-\frac{r}{2}} + t, \quad r \in (1, 2], \quad k = 2, 3,$$

$a_{k,r}$ ($r \in (1, 2], k = 2, 3$) being positive constants that can be explicitly determined from the data.

Proof. We first consider the case when $l = 3$ and $p \in (1, 2)$. Even if we limit ourselves to the case when $M_3 \geq 0$, our techniques can also be used, with minor changes, in the case when $M_3 < 0$.

We fix $\delta, t, R > 0$ and let η_R be the cut-off function defined at the beginning of the proof of Theorem 3.3. For any $\alpha, \beta > 0$ and any $f \in C_b^2(\mathbb{R}^N)$ we define the function $g_\delta : [0, t] \rightarrow C(B(R))$ by

$$(4.17) \quad g_\delta(s) = T_R(s) \left(\left(\alpha |T_R(t-s)f|^2 + \beta \eta_R^2 |DT_R(t-s)f|^2 + \eta_R^4 |D^2 T_R(t-s)f|^2 + \delta \right)^{\frac{p}{2}} - \delta^{\frac{p}{2}} \right), \quad 0 \leq s \leq t,$$

where $T_R(\cdot)$ is the semigroup generated in $C_b(\overline{B}(R))$ by the realization A_R of the operator \mathcal{A} with Dirichlet boundary conditions (see Section 2). To simplify the notation, throughout the remainder of the proof, we set $\phi_R^r := T_R(t-r)f$ and $\phi^r := T(t-r)f$ for any $r \in [0, t]$.

As is easily seen the function

$$(\alpha |\phi_R^r|^2 + \beta \eta_R^2 |D\phi_R^r|^2 + \eta_R^4 |D^2 \phi_R^r|^2 + \delta)^{p/2} - \delta^{p/2}$$

belongs to $D(A_R)$ for any $r \in [0, t)$. Recalling that $\mathcal{A}(\delta^{p/2}) = 0$, we get

$$\begin{aligned} g'_\delta(s) = pT_R(s) & \left\{ \left[\alpha \sum_{i,j=1}^N q_{ij} D_i \phi_R^s D_j \phi_R^s + \beta \eta_R^2 \sum_{i,j,h=1}^N q_{ij} D_{ih} \phi_R^s D_{jh} \phi_R^s \right. \right. \\ & + \eta_R^4 \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} \phi_R^s D_{jhk} \phi_R^s + \frac{\beta}{2} \mathcal{A}(\eta_R^2) |D\phi_R^s|^2 \\ & + \frac{1}{2} \mathcal{A}(\eta_R^4) |D^2 \phi_R^s|^2 + 4\beta \eta_R \sum_{i,j,h=1}^N q_{ij} D_i \eta_R D_h \phi_R^s D_{jh} \phi_R^s \\ & + 8\eta_R^3 \sum_{i,j,h,k=1}^N q_{ij} D_j \eta_R D_{hk} \phi_R^s D_{ihk} \phi_R^s \\ & \left. - \beta \eta_R^2 \left(\sum_{i,j,h=1}^N D_h q_{ij} D_h \phi_R^s D_{ij} \phi_R^s + \sum_{j,h=1}^N D_h b_j D_j \phi_R^s D_h \phi_R^s \right) \right\} \end{aligned}$$

$$\begin{aligned}
& -\eta_R^4 \left(\sum_{i,j,h,k=1}^N (2D_h q_{ij} D_{ijk} \phi_R^s + D_{hk} q_{ij} D_{ij} \phi_R^s) D_{hk} \phi_R^s \right. \\
& \quad \left. + \sum_{j,h,k=1}^N (2D_h b_j D_{jk} \phi_R^s + D_{hk} b_j D_j \phi_R^s) D_{hk} \phi_R^s \right) \Bigg] \\
& \times \left(\alpha |\phi_R^s|^2 + \beta \eta_R^2 |D \phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 + \delta \right)^{\frac{p}{2}-1} \Bigg\} \\
& - \frac{p(2-p)}{4} T_R(s) \left[\sum_{i,j=1}^N q_{ij} D_i \left(\alpha |\phi_R^s|^2 + \beta \eta_R^2 |D \phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 \right) \right. \\
& \quad \times D_j \left(\alpha |\phi_R^s|^2 + \beta \eta_R^2 |D \phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 \right) \\
& \quad \left. \times \left(\alpha |\phi_R^s|^2 + \beta \eta_R^2 |D \phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 + \delta \right)^{\frac{p}{2}-2} \right].
\end{aligned}$$

Arguing as in the proof of (4.4) and recalling the inequality $(a+b)^2 \leq (1+\varepsilon)a^2 + (1+\varepsilon^{-1})b^2$ for any $a, b, \varepsilon > 0$, we easily deduce that

$$\begin{aligned}
& \frac{1}{4} \sum_{i,j=1}^N q_{ij} D_i \left(\alpha (\phi_R^s)^2 + \beta \eta_R^2 |D \phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 \right) \\
& \quad \times D_j \left(\alpha (\phi_R^s)^2 + \beta \eta_R^2 |D \phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 \right) \\
& \leq (1+\varepsilon) \left[\alpha |\phi_R^s| \left(\sum_{i,j=1}^N q_{ij} D_i \phi_R^s D_j \phi_R^s \right)^{\frac{1}{2}} + \beta \eta_R^2 |D \phi_R^s| \left(\sum_{i,j,h=1}^N q_{ij} D_{ih} \phi_R^s D_{jh} \phi_R^s \right)^{\frac{1}{2}} \right. \\
& \quad \left. + \eta_R^4 |D^2 \phi_R^s| \left(\sum_{i,j,h,k=1}^N q_{ij} D_{ihk} \phi_R^s D_{jhk} \phi_R^s \right)^{\frac{1}{2}} \right]^2 \\
& \quad + \left(1 + \frac{1}{\varepsilon} \right) \left(\beta \eta_R |D \phi_R^s|^2 + 2\eta_R^3 |D^2 \phi_R^s|^2 \right)^2 \sum_{i,j=1}^N q_{ij} D_i \eta_R D_j \eta_R \\
& \leq (1+\varepsilon) \left[\alpha \sum_{i,j=1}^N q_{ij} D_i \phi_R^s D_j \phi_R^s + \beta \eta_R^2 \sum_{i,j,h=1}^N q_{ij} D_{ih} \phi_R^s D_{jh} \phi_R^s \right. \\
& \quad \left. + \eta_R^4 \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} \phi_R^s D_{jhk} \phi_R^s \right]
\end{aligned}$$

$$\begin{aligned}
& \times (\alpha|\phi_R^s|^2 + \beta\eta_R^2|D\phi_R^s|^2 + \eta_R^4|D^2\phi_R^s|^2) \\
& + \left(1 + \frac{1}{\varepsilon}\right) (\beta|D\phi_R^s|^2 + 4\eta_R^2|D^2\phi_R^s|^2) (\beta\eta_R^2|D\phi_R^s|^2 + \eta_R^4|D^2\phi_R^s|^2) \\
& \times \sum_{i,j=1}^N q_{ij} D_i \eta_R D_j \eta_R.
\end{aligned}$$

Hence,

$$\begin{aligned}
g'_\delta(s) \geq pT_R(s) & \left\{ \left[\alpha C_{\varepsilon,p} \sum_{i,j=1}^N q_{ij} D_i \phi_R^s D_j \phi_R^s + \beta C_{\varepsilon,p} \eta_R^2 \sum_{i,j,h=1}^N q_{ij} D_{ih} \phi_R^s D_{jh} \phi_R^s \right. \right. \\
& + C_{\varepsilon,p} \eta_R^4 \sum_{i,j,h,k=1}^N q_{ij} D_{ihk} \phi_R^s D_{jhk} \phi_R^s + \frac{\beta}{2} \mathcal{A}(\eta_R^2) |D\phi_R^s|^2 \\
& + \frac{1}{2} \mathcal{A}(\eta_R^4) |D^2\phi_R^s|^2 + 4\beta\eta_R \sum_{i,j,h=1}^N q_{ij} D_i \eta_R D_h \phi_R^s D_{jh} \phi_R^s \\
& + 8\eta_R^3 \sum_{i,j,h,k=1}^N q_{ij} D_j \eta_R D_{hk} \phi_R^s D_{ihk} \phi_R^s \\
& - \beta\eta_R^2 \left(\sum_{i,j,h=1}^N D_h q_{ij} D_h \phi_R^s D_{ij} \phi_R^s + \sum_{j,h=1}^N D_h b_j D_j \phi_R^s D_h \phi_R^s \right) \\
& - \eta_R^4 \left(\sum_{i,j,h,k=1}^N (2D_h q_{ij} D_{ijk} \phi_R^s + D_{hk} q_{ij} D_{ij} \phi_R^s) D_{hk} \phi_R^s \right. \\
& \quad \left. + \sum_{j,h,k=1}^N (2D_h b_j D_{jk} \phi_R^s + D_{hk} b_j D_j \phi_R^s) D_{hk} \phi_R^s \right) \\
& - (2-p) \left(1 + \frac{1}{\varepsilon}\right) (\beta|D\phi_R^s|^2 + 4\eta_R^2|D^2\phi_R^s|^2) \\
& \quad \left. \times \sum_{i,j=1}^N q_{ij} D_i \eta_R D_j \eta_R \right] \\
& \times (\alpha|\phi_R^s|^2 + \beta\eta_R^2|D\phi_R^s|^2 + \eta_R^4|D^2\phi_R^s|^2 + \delta)^{\frac{p}{2}-1} \Bigg\},
\end{aligned}$$

where $C_{\varepsilon,p} = 1 + (1 + \varepsilon)(p - 2)$. Taking assumption $H3$ into account and arguing as in the proof of (3.18), one can easily check that

$$(4.18) \quad \sum_{i,j=1}^N q_{ij}(x) D_i \eta_R(x) D_j \eta_R(x) \leq C''' \nu(x),$$

for any $x \in B(R)$ and some positive constant C''' , independent of R . Moreover, in view of (3.18)-(3.20) we can write

$$(4.19) \quad \mathcal{A}(\eta_R^2) \geq -2(C' + C'')\nu,$$

$$(4.20)$$

$$\mathcal{A}(\eta_R^4) = 2\eta_R^2 \mathcal{A}(\eta_R^2) + 2 \sum_{i,j=1}^N q_{ij} D_i \eta_R^2 D_j \eta_R^2 \geq 2\eta_R^2 \mathcal{A}(\eta_R^2) \geq -4(C' + C'')\eta_R^2 \nu,$$

$$(4.21) \quad \left| \eta_R \sum_{i,j,h=1}^N q_{ij} D_i \eta_R D_h \phi_R^s D_{jh} \phi_R^s \right| \leq \frac{C'' N}{4\varepsilon} \nu |D\phi_R^s|^2 + \varepsilon C'' \eta_R^2 \nu |D^2 \phi_R^s|^2,$$

$$(4.22)$$

$$\left| \eta_R^3 \sum_{i,j,h,k=1}^N q_{ij} D_j \eta_R D_{hk} \phi_R^s D_{ihk} \phi_R^s \right| \leq \frac{C'' N}{4\varepsilon} \eta_R^2 \nu |D^2 \phi_R^s|^2 + \varepsilon C'' \eta_R^4 \nu |D^3 \phi_R^s|^2,$$

for any $\varepsilon > 0$.

Now, from assumptions $H1$, $H4-1-\gamma$ and (4.18), (4.19)-(4.22), we easily deduce that

$$\begin{aligned} g'_\delta(s) \geq pT_R(s) \Bigg\{ & \left[\left(\alpha C_{\varepsilon,p} - \beta(C' + C'') - \beta \frac{C'' N}{\varepsilon} + \beta(p-2)C''' \left(1 + \frac{1}{\varepsilon}\right) \right) \nu \right. \\ & - \beta \frac{CN^2}{4\varepsilon} \nu^\gamma - \eta_R^2 \left(\beta d + r \frac{N^2}{4\varepsilon} \right) |D\phi_R^s|^2 \\ & + \left(\left(\beta(C_{\varepsilon,p} - 4\varepsilon C'') - 2(C' + C'') - \frac{2C'' N}{\varepsilon} \right. \right. \\ & \quad \left. \left. + 4(p-2)C''' \left(1 + \frac{1}{\varepsilon}\right) \right) \nu \right. \\ & \quad \left. - \left(\beta C_\varepsilon N + \frac{CN^2}{2\varepsilon} + K_1^+ \right) \nu^\gamma - \eta_R^2 (2d + r\varepsilon N) \right) \eta_R^2 |D^2 \phi_R^s|^2 \\ & \left. + ((C_{\varepsilon,p} - 8\varepsilon C'') \nu - 2C_\varepsilon N \nu^\gamma) \eta_R^4 |D^3 \phi_R^s|^2 \right] \\ & \times (\alpha |\phi_R^s|^2 + \beta \eta_R^2 |D\phi_R^s|^2 + \eta_R^4 |D^2 \phi_R^s|^2 + \delta)^{\frac{p}{2}-1} \Bigg\}. \end{aligned}$$

In view of assumption $H4-3-\gamma$, we have

$$\beta d(x) + r(x) \frac{N^2}{4\varepsilon} \leq \left(\beta - \frac{N^2}{4\varepsilon} L_3^{-1} \right) M_3^+(\nu(x))^\gamma + \frac{N^2}{4\varepsilon} L_3^{-1} M_3^+(\nu(x))^\gamma,$$

$$2d(x) + \varepsilon r(x) N \leq 2M_3^+(\nu(x))^\gamma,$$

for any $x \in \mathbb{R}^N$, provided $\varepsilon < 2L_3N^{-1}$ and $\beta > N^2/(4\varepsilon L_3)$. Here L_3 is as in assumption H_4 - β - γ and M_3 is given by (4.1).

We now choose ε sufficiently small and $\alpha = \alpha_p$, $\beta = \beta_p$ sufficiently large such that

$$\left\{ \begin{array}{l} 0 < A_1 := \alpha(1 + (1 + \varepsilon)(p - 2)) - \beta(C' + C'') \\ \quad - \beta C'' N \varepsilon^{-1} + \beta(p - 2)C'''(1 + \varepsilon^{-1}), \\ 0 < A_2 := \beta(1 + (1 + \varepsilon)(p - 2) - 4\varepsilon C'') - 2(C' + C'') \\ \quad - 2C'' N \varepsilon^{-1} + 4(p - 2)C'''(1 + \varepsilon^{-1}), \\ (\beta C N^2 \varepsilon^{-1} + (4\beta - N^2 \varepsilon^{-1} L_3^{-1})M_3^+ + N^2 \varepsilon^{-1} L_3^{-1} M_3^+) \nu_0^{\gamma-1} < 4A_1, \\ (2\beta C \varepsilon N + C N^2 \varepsilon^{-1} + 2K_1^+ + 4M_3^+) \nu_0^{\gamma-1} < 2A_2, \\ 0 < A_3 := 1 + (p - 2)(1 + \varepsilon) - 8\varepsilon C'', \quad 2C \varepsilon N \nu_0^{\gamma-1} < A_3, \\ \varepsilon < 2L_3 N^{-1}, \quad 4\beta > N^2 L_3^{-1} \varepsilon^{-1}. \end{array} \right.$$

Consequently, we can find out a positive constant c_p such that

$$(4.23) \quad g'_\delta(s) \geq c_p T_R(s) \left(\eta_R^4 (|D\phi_R^s|^2 + |D^2\phi_R^s|^2 + |D^3\phi_R^s|^2) \right. \\ \left. \times \left(\alpha_p |\phi_R^s|^2 + \beta_p \eta_R^2 |D\phi_R^s|^2 + \eta_R^4 |D^2\phi_R^s|^2 + \delta \right)^{\frac{p}{2}-1} \right).$$

Then, integrating (4.23) in $[\varepsilon, t - \varepsilon]$ ($\varepsilon > 0$) we get

$$(4.24) \quad c_p \int_\varepsilon^{t-\varepsilon} \left(T_R(s) \left(\eta_R^4 (|D\phi_R^s|^2 + |D^2\phi_R^s|^2 + |D^3\phi_R^s|^2) \right. \right. \\ \left. \left. \times (\alpha_p |\phi_R^s|^2 + \beta_p \eta_R^2 |D\phi_R^s|^2 + \eta_R^4 |D^2\phi_R^s|^2 + \delta)^{\frac{p}{2}-1} \right) \right) (x) ds \\ \leq \left(T_R(t - \varepsilon) \left((\alpha_p |T_R(\varepsilon)f|^2 + \beta_p \eta_R^2 |DT_R(\varepsilon)f|^2 + \eta_R^4 |D^2T_R(\varepsilon)f|^2 + \delta)^{\frac{p}{2}} - \delta^{\frac{p}{2}} \right) \right) (x) \\ - \left(T_R(\varepsilon) \left((\alpha_p |\phi_R^\varepsilon|^2 + \beta_p \eta_R^2 |D\phi_R^\varepsilon|^2 + \eta_R^4 |D^2\phi_R^\varepsilon|^2 + \delta)^{\frac{p}{2}} - \delta^{\frac{p}{2}} \right) \right) (x) \\ \leq \left(T_R(t - \varepsilon) \left((\alpha_p |T_R(\varepsilon)f|^2 + \beta_p \eta_R^2 |DT_R(\varepsilon)f|^2 + \eta_R^4 |D^2T_R(\varepsilon)f|^2)^{\frac{p}{2}} \right) \right) (x).$$

Therefore, taking (2.11) into account, from Lemma 4.6 we have

$$\lim_{R \rightarrow +\infty} \left\{ T_R(s) \left[\eta_R^4 (|D\phi_R^s|^2 + |D^2\phi_R^s|^2 + |D^3\phi_R^s|^2) \right. \right. \\ \left. \left. \times (\alpha_p |\phi_R^s|^2 + \beta_p \eta_R^2 |D\phi_R^s|^2 + \eta_R^4 |D^2\phi_R^s|^2 + \delta)^{\frac{p}{2}-1} \right] \right\} (x) \\ = \left\{ T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \right. \\ \left. \left. \times (\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}-1} \right] \right\} (x),$$

the convergence being uniform in $[\varepsilon, t - \varepsilon] \times B(k)$ for any $k > 0$, and

$$\lim_{R \rightarrow +\infty} \left(T_R(t - \varepsilon) \left((\alpha_p |T_R(\varepsilon)f|^2 + \beta_p \eta_R^2 |DT_R(\varepsilon)f|^2 + \eta_R^4 |D^2T_R(\varepsilon)f|^2)^{\frac{p}{2}} \right) \right) (x) \\ = \left(T(t - \varepsilon) \left((\alpha_p |T(\varepsilon)f|^2 + \beta_p |DT(\varepsilon)f|^2 + |D^2T(\varepsilon)f|^2)^{\frac{p}{2}} \right) \right) (x),$$

uniformly for $s \in [\varepsilon, t - \varepsilon]$. Then, from (4.24) we get

$$\begin{aligned}
 (4.25) \quad & c_p \int_{\varepsilon}^{t-\varepsilon} \left\{ T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \right. \\
 & \quad \left. \left. \times (\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}-1} \right] \right\} (x) ds \\
 & \leq \left(T(t-\varepsilon)(\alpha_p |T(\varepsilon)f|^2 + \beta_p |DT(\varepsilon)f|^2 + |D^2T(\varepsilon)f|^2)^{\frac{p}{2}} \right) (x),
 \end{aligned}$$

for any $x \in \mathbb{R}^N$. Now, from Lemma 4.2 it follows that

$$\lim_{\varepsilon \rightarrow 0^+} D^j T(\varepsilon)f = D^j f, \quad j = 0, 1, 2,$$

uniformly in $B(k)$ for all $k > 0$. Therefore from Lemma 4.6 we have

$$\begin{aligned}
 (4.26) \quad & \lim_{\varepsilon \rightarrow 0^+} \left(T(t-\varepsilon)(\alpha_p |T(\varepsilon)f|^2 + \beta_p |DT(\varepsilon)f|^2 + |D^2T(\varepsilon)f|^2)^{\frac{p}{2}} \right) (x) \\
 & = \left(T(t)((\alpha_p |f|^2 + \beta_p |Df|^2 + |D^2f|^2)^{\frac{p}{2}}) \right) (x),
 \end{aligned}$$

for any $x \in \mathbb{R}^N$. Hence, taking the limit as ε tends to 0 in (4.25), we get

$$\begin{aligned}
 (4.27) \quad & c_p \int_0^t T(s) \left((|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \\
 & \quad \left. \times (\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}-1} \right) (x) ds \\
 & \leq \left(T(t)(\alpha_p |f|^2 + \beta_p |Df|^2 + |D^2f|^2)^{\frac{p}{2}} \right) (x),
 \end{aligned}$$

for any $x \in \mathbb{R}^N$.

We now adapt the technique in [4, Proposition 4.3] to our situation. Let us observe that from (4.9) (with $l = 2$) and the Young and Hölder inequalities, we deduce that for any $\gamma \in \mathbb{R}$,

$$\begin{aligned}
 & (|DT(t)f|^2 + |D^2T(t)f|^2 + |D^3T(t)f|^2)^{\frac{p}{2}} \\
 & = (|DT(s)(\phi^s)|^2 + |D^2T(s)(\phi^s)|^2 + |D^3T(s)(\phi^s)|^2)^{\frac{p}{2}} \\
 & \leq e^{\sigma_{3,p} s} T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2)^{\frac{p}{2}} \right] + e^{\sigma_{3,p} s} T(s) (|\phi^s|^p) \\
 & \leq e^{\sigma_{3,p} s} T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2)^{\frac{p}{2}} \right. \\
 & \quad \times (\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{-\gamma} \\
 & \quad \left. \times (\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\gamma} \right] \\
 & + e^{\sigma_{3,p} s} T(s) (|f|^p) \\
 & \leq e^{\sigma_{3,p} s} \left\{ T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \right. \\
 & \quad \left. \left. \times (\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{-\frac{2\gamma}{p}} \right] \right\}^{\frac{p}{2}} \\
 & \quad \times \left\{ T(s) \left[(\alpha_p |\phi^s|^2 + \beta_p |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{2\gamma}{2-p}} \right] \right\}^{1-\frac{p}{2}} \\
 & + e^{\sigma_{3,p} s} T(s) (|f|^p).
 \end{aligned}$$

Choosing $\gamma = p(2 - p)/4$, using Jensen and Young inequalities, and recalling that

$$\begin{aligned} & (|\phi^s|^2 + |D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}} \\ & \leq e^{\sigma_{2,p}(t-s)}T(t-s)((f^2 + |Df|^2 + |D^2f|^2 + \delta)^{\frac{p}{2}}) + \delta^{\frac{p}{2}} \end{aligned}$$

(see again (4.8)), we get for all $\varepsilon > 0$

$$\begin{aligned} (4.28) \quad & (|DT(t)f|^2 + |D^2T(t)f|^2 + |D^3T(t)f|^2)^{\frac{p}{2}} \\ & \leq e^{\sigma_{3,p}s} \left\{ T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \right. \\ & \quad \left. \left. \times (\alpha_p|\phi^s|^2 + \beta_p|D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}-1} \right] \right\}^{\frac{p}{2}} \\ & \quad \times \left\{ T(s) \left[(\alpha_p|\phi^s|^2 + \beta_p|D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}} \right] \right\}^{1-\frac{p}{2}} + e^{\sigma_{3,p}s}T(t)(|f|^p) \\ & \leq e^{\sigma_{3,p}s} \left\{ \frac{p}{2}\varepsilon^{\frac{2}{p}}T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \right. \\ & \quad \left. \left. \times (\alpha_p|\phi^s|^2 + \beta_p|D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}-1} \right] \right. \\ & \quad \left. + \left(1 - \frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}T(s) \left[(\alpha_p|\phi^s|^2 + \beta_p|D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}} \right] \right\} \\ & \quad + e^{\sigma_{3,p}s}T(t)(|f|^p) \\ & \leq e^{\sigma_{3,p}s} \left\{ \frac{p}{2}\varepsilon^{\frac{2}{p}}T(s) \left[(|D\phi^s|^2 + |D^2\phi^s|^2 + |D^3\phi^s|^2) \right. \right. \\ & \quad \left. \left. \times (\alpha_p|\phi^s|^2 + \beta_p|D\phi^s|^2 + |D^2\phi^s|^2 + \delta)^{\frac{p}{2}-1} \right] \right. \\ & \quad + \left(1 - \frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\max\{\alpha_p, \beta_p, 1\}^{\frac{p}{2}} \\ & \quad \left. \times \left[e^{\sigma_{2,p}(t-s)}T(t) \left((f^2 + |Df|^2 + |D^2f|^2 + \delta)^{\frac{p}{2}} \right) + \delta^{\frac{p}{2}} \right] \right. \\ & \quad \left. + T(t)(|f|^p) \right\}. \end{aligned}$$

Multiplying the first and last side of (4.28) by $e^{-\sigma_{3,p}s}$, integrating in $(0, t)$, pointwise with respect to $x \in \mathbb{R}^N$, and taking (4.27) into account, we get

$$\begin{aligned} (4.29) \quad & \frac{1 - e^{-\sigma_{3,p}t}}{\sigma_{3,p}} (|DT(t)f|^2 + |D^2T(t)f|^2 + |D^3T(t)f|^2)^{\frac{p}{2}} \\ & \leq \frac{p}{2c_p}\varepsilon^{\frac{2}{p}}T(t)((\alpha_p f^2 + \beta_p|Df|^2 + |D^2f|^2)^{\frac{p}{2}}) + tT(t)(|f|^p) \\ & \quad + \left(1 - \frac{p}{2}\right)\varepsilon^{\frac{2}{p-2}}\max\{\alpha_p, \beta_p, 1\}^{\frac{p}{2}} \\ & \quad \times \left[\frac{e^{\sigma_{2,p}t} - 1}{\sigma_{2,p}}T(t)((f^2 + |Df|^2 + |D^2f|^2 + \delta)^{\frac{p}{2}}) + \delta^{\frac{p}{2}} \right]. \end{aligned}$$

Letting δ tend to 0 in (4.29) gives

$$\begin{aligned} & \frac{1 - e^{-\sigma_{3,p}t}}{\sigma_{3,p}} (|DT(t)f|^2 + |D^2T(t)f|^2 + |D^3T(t)f|^2)^{\frac{p}{2}} \\ & \leq \max\{\alpha_p, \beta_p, 1\}^{\frac{p}{2}} \left\{ \min_{\varepsilon > 0} \left(\frac{p}{2c_p} \varepsilon^{\frac{2}{p}} + \left(1 - \frac{p}{2}\right) \frac{e^{\sigma_{2,p}t} - 1}{\sigma_{2,p}} \varepsilon^{\frac{2}{p-2}} \right) + t \right\} \\ & \quad \times T(t) (f^2 + |Df|^2 + |D^2f|^2)^{\frac{p}{2}} \\ & = \max\{\alpha_p, \beta_p, 1\}^{\frac{p}{2}} \left[c_p^{-\frac{p}{2}} \left(\frac{e^{\sigma_{2,p}t} - 1}{\sigma_{2,p}} \right)^{1-\frac{p}{2}} + t \right] T(t) (f^2 + |Df|^2 + |D^2f|^2)^{\frac{p}{2}} \end{aligned}$$

and (4.16) follows with $p \in (1, 2)$.

The case when $p = 2$ is similar and even simpler. It suffices to apply the previous arguments to the function g_0 defined by (4.17) (with $\delta = 0$ and $p = 2$).

In the case when $p > 2$, (4.16) (with $l = 3$) can be obtained from the case when $p = 2$ observing that

$$\begin{aligned} |(D^3T(t)f)(x)|^p &= (|(D^3T(t)f)(x)|^2)^{\frac{p}{2}} \\ &\leq \left(\frac{\sigma_{3,2}}{(1 - e^{-\sigma_{3,2}t})} \phi_{3,2}(t) \right)^{\frac{p}{2}} ((T(t)(f^2 + |Df|^2 + |D^2f|^2))(x))^{\frac{p}{2}} \end{aligned}$$

and then applying Hölder's inequality.

The proof of (4.16) in the case when $l = 1, 2$ is completely similar, hence it is omitted. Actually, one can prove that

$$\begin{aligned} & \left(\sum_{j=0}^l |(D^jT(t)f)(x)|^2 \right)^{\frac{p}{2}} \\ (4.30) \quad & \leq \left(\frac{\sigma_{l,p \wedge 2}}{1 - e^{-\sigma_{l,p \wedge 2}t}} \phi_{l,p \wedge 2}(t) \right)^{\max\{1, \frac{p}{2}\}} \left(T(t) \left(\sum_{j=0}^{l-1} |D^j f|^2 \right)^{\frac{p}{2}} \right)(x), \end{aligned}$$

for any $t > 0$, $x \in \mathbb{R}^N$, $p \in (1, +\infty)$. \square

Corollary 4.8. *Let $l \in \{2, 3\}$ and let assumptions H1-H3, H4-l- γ be satisfied. Then, for any $f \in C_b(\mathbb{R}^N)$ and any $p \in (1, +\infty)$ we have*

$$(4.31) \quad |(D^lT(t)f)(x)|^p \leq \left(\sum_{k=1}^l \prod_{j=0}^{k-1} \frac{\hat{\sigma}_{l-j,p \wedge 2}}{1 - e^{-\hat{\sigma}_{l-j,p \wedge 2}t/l}} \phi_{l-j,p \wedge 2}(t/l) \right)^{\max\{1, \frac{p}{2}\}} (T(t)(|f|^p))(x),$$

for any $t > 0$ and any $x \in \mathbb{R}^N$, where $\phi_{j,r}$ and $\hat{\sigma}_{j,r}$ are as in Theorem 4.7.

Proof. We begin by proving (4.31) in the case when $l = 3$ and $p \leq 2$. We fix $f \in C_b(\mathbb{R}^N)$ and $t > 0$. Splitting $T(t)f = T(t/3)T(2t/3)f$ and applying (4.16) with f being replaced by $T(2t/3)f$, we get

$$\begin{aligned} |(D^3T(t)f)(x)|^p &\leq \frac{\hat{\sigma}_{3,p}}{1 - e^{-\hat{\sigma}_{3,p}t/3}} \phi_{3,p}(t/3) \\ (4.32) \quad & \times T(t/3) \left\{ (|T(2t/3)f|^2 + |DT(2t/3)f|^2 + |D^2T(2t/3)f|^2)^{\frac{p}{2}} \right\}(x), \end{aligned}$$

for any $x \in \mathbb{R}^N$. Now, taking (4.30) into account, with $T(t)f$ replaced by $T(2t/3)f$, we get

$$\begin{aligned}
 (4.33) \quad & (|T(2t/3)f|^2 + |DT(2t/3)f|^2 + |D^2T(2t/3)f|^2)^{\frac{p}{2}} \\
 & \leq \frac{\hat{\sigma}_{2,p}}{1 - e^{-\hat{\sigma}_{2,p}t/3}} \phi_{2,p}(t/3) T(t/3) \left\{ (|T(t/3)f|^2 + |DT(t/3)f|^2)^{\frac{p}{2}} \right\} + T(2t/3)(|f|^p) \\
 & \leq \frac{\hat{\sigma}_{2,p}}{1 - e^{-\hat{\sigma}_{2,p}t/3}} \phi_{2,p}(t/3) \left[\frac{\hat{\sigma}_{1,p}}{1 - e^{-\hat{\sigma}_{1,p}t/3}} \phi_{1,p}(t/3) T(2t/3)(|f|^p + T(2t/3)(|f|^p)) \right] \\
 & \quad + T(2t/3)(|f|^p).
 \end{aligned}$$

Combining (4.32) and (4.33) gives (4.31) for $p \leq 2$. If $p > 2$ and $l = 3$, (4.31) follows from the case when $p = 2$ applying Hölder's inequality. The proof of (4.31) (with $l = 2$) is similar: we write $|(D^2T(t)f)(x)|^2 = |(D^2T(t/2)T(t/2)f)(x)|^2$ and apply (4.16), respectively with $l = 1$ and $l = 2$, to $T(t/2)f$. We omit the details. \square

We now prove a Liouville-type theorem. Namely, we show that under suitable assumptions on the coefficients, if $u \in C_b(\mathbb{R}^N)$ satisfies the equation $\mathcal{A}u = 0$, then u is constant. As already stressed in the Introduction, such a result fails in general. We refer the reader to [18] for further details.

Theorem 4.9. *Suppose that assumptions H1-H3 and H4-1- γ ($\gamma \in (0, 1)$) hold true. Further assume that*

$$(4.34) \quad \frac{C^2 N^3 \nu_0^{\gamma-1}}{4} + M_1 < 0,$$

where M_1 is given by (4.1) and C is as in H4-1- γ . If $f \in C_b(\mathbb{R}^N)$ satisfies $\mathcal{A}f = 0$, then f is constant.

Proof. We observe that if $f \in C_b(\mathbb{R}^N)$ is such that $\mathcal{A}f = 0$, then, by local elliptic regularity, it follows that $f \in D_{\max}(\mathcal{A})$ (see (3.35)). Moreover, the function $t \mapsto (T(t)f)(x)$ is continuously differentiable in $(0, +\infty)$ and $D_t((T(t)f)(x)) = (\mathcal{A}T(t)f)(x) = (T(t)\mathcal{A}f)(x)$ (see e.g. [12, Theorem 4.1]). Hence, $D_tT(t)f = 0$ for any $t > 0$, so that $T(t)f = f$ for any $t > 0$. From (4.16) we deduce that

$$|(DT(t)f)(x)| \leq \frac{\tilde{C}}{\sqrt{t}} e^{\omega t} ((T(t)f^2)(x))^{1/2} \leq \frac{\tilde{C}}{\sqrt{t}} e^{\omega t} \|f\|_{\infty}, \quad \forall t > 0, \quad x \in \mathbb{R}^N,$$

for some $\tilde{C} > 0$ and some $\omega < 0$. Indeed, (4.34) implies that $\hat{\sigma}_{1,2} = 0$. As t tends to $+\infty$, we get

$$Df(x) = \lim_{t \rightarrow +\infty} (DT(t)f)(x) = 0, \quad \forall x \in \mathbb{R}^N,$$

so that f is constant. \square

4.1. Further estimates for $\mathcal{A} = \Delta + \sum_{j=1}^N b_j(x)D_j$. We now consider the particular case when $q_{ij} = \delta_{ij}$. It is well known that in this case the gradient estimate (4.2) (with $l = 1$) can also be proved for $p = 1$. More precisely, if there exists a constant $d_0 \in \mathbb{R}$ such that $\sum_{i,j=1}^N D_i b_j(x) \xi_i \xi_j \leq d_0 |\xi|^2$ for any $x, \xi \in \mathbb{R}^N$, then we have

$$|(DT(t)f)(x)| \leq e^{d_0 t} T(t)(|Df|)(x), \quad \forall t > 0, \quad x \in \mathbb{R}^N,$$

for any $f \in C_b^1(\mathbb{R}^N)$ (see e.g. [1, 2, 4]).

With this in mind, here we extend the estimates for the second- and third-order derivatives given in Theorem 4.7 to the case $p = 1$.

Theorem 4.10. *Let $l \in \{2, 3\}$ and let assumptions $H2$, $H4$ - l - γ (with $q_{ij} = \delta_{ij}$) and (3.4) be satisfied. Then,*

$$(4.35) \quad |(D^l T(t)f)(x)| \leq e^{\sigma_l t} \left(T(t) \left[\left(\sum_{j=1}^l |D^j f|^2 \right)^{\frac{1}{2}} \right] (x) \right), \quad \forall t > 0, x \in \mathbb{R}^N,$$

where $\sigma_l = M_l + (l-1)M_l^+$ (see (4.1)).

Proof. We limit ourselves to sketching the proof in the case when $l = 2$.

Let $\delta > 0$, $f \in C_b^1(\mathbb{R}^N)$ and consider the function $w_\delta : \mathbb{R}_+ \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined by

$$w_\delta(t, x) = (|Du(t, x)|^2 + |D^2 u(t, x)|^2 + \delta)^{\frac{1}{2}},$$

where, as usual, $u(t, \cdot) = T(t)f$. Arguing as in the proof of Theorem 4.3 and taking Cauchy-Schwarz inequality into account, we can easily show that w_δ turns out to solve the Cauchy problem

$$\begin{cases} D_t w_\delta(t, x) = \mathcal{A} w_\delta(t, x) + g_\delta(t, x), & t > 0, x \in \mathbb{R}^N, \\ w_\delta(0, x) = (|Df(x)|^2 + |D^2 f(x)|^2 + \delta)^{\frac{1}{2}}, & x \in \mathbb{R}^N, \end{cases}$$

where

$$g_\delta \leq w_\delta^{-1} \left(\sum_{j,h=1}^N D_h b_j D_j u D_h u + 2 \sum_{j,h,k=1}^N D_h b_j D_{jk} u D_{hk} u + \sum_{j,h,k=1}^N D_{hk} b_j D_j u D_{hk} u \right).$$

From $H4$ -2- γ we get for all $\varepsilon > 0$

$$g_\delta \leq w_\delta^{-1} \left(\left(d + \frac{N^2}{4\varepsilon} r \right) |Du|^2 + 2 \left(d + \frac{\varepsilon N}{2} r \right) |D^2 u|^2 \right).$$

We choose $\varepsilon = \sqrt{N/2}$ so that $\varepsilon N/2 = N^2/(4\varepsilon) = N^{3/2}/\sqrt{8}$ and we get

$$d(x) + \frac{N^2}{4\varepsilon} r(x) \leq M_2, \quad d(x) + \frac{\varepsilon N}{2} r(x) \leq M_2, \quad \forall x \in \mathbb{R}^N.$$

Now, if $M_2 \geq 0$ it follows that $g_\delta \leq 2M_2 w_\delta$; if $M_2 < 0$ we have

$$g_\delta \leq M_2 (|Du|^2 + |D^2 u|^2 + \delta)^{-\frac{1}{2}} (|Du|^2 + |D^2 u|^2) \leq M_2 w_\delta - M_2 \delta^{\frac{1}{2}}.$$

Following step by step the proofs of Theorems 4.3 and 4.4, we can easily get (4.35). \square

Remark 4.11. It is well known that in general estimates (4.16) and (4.31) fail for $p = 1$. We refer the reader to [4, Example 5.5] where a counterexample is exhibited in the case when $\mathcal{A} = \Delta$.

5. INVARIANT MEASURES

In this section we will show that the pointwise estimates of the previous section imply global estimates for the derivatives of $T(t)f$ in L^p -spaces associated with the semigroup invariant measure, whenever it exists.

We recall that a probability measure μ in \mathbb{R}^N is an invariant measure of the semigroup $\{T(t)\}_{t \geq 0}$ if

$$\int_{\mathbb{R}^N} (T(t)f)(x) d\mu(x) = \int_{\mathbb{R}^N} f(x) d\mu(x),$$

for any $f \in C_b(\mathbb{R}^N)$ and any $t > 0$. If an invariant measure exists, then it is unique, it is absolutely continuous with respect to the Lebesgue measure and it has a strictly positive density (see for example [13, Section 6]).

The Has'minskii theorem shows that if there exists a nonnegative function $\varphi \in C^2(\mathbb{R}^N)$ such that $\lim_{|x| \rightarrow +\infty} \mathcal{A}\varphi(x) = -\infty$, then there exists an invariant measure of the semigroup $\{T(t)\}_{t \geq 0}$. See [8, Chapter III, Theorem 5.1] for more details.

If an invariant measure μ exists, then the semigroup $\{T(t)\}_{t \geq 0}$ can be extended to a strongly continuous contraction semigroup in L^p_μ , which, in general, is not analytic (see [14]).

The following theorem both extends the results in [4, Corollary 5.4] to our situation, and gives, for any $p \in (1, +\infty)$, L^p -estimates for the second- and third-order derivatives of $T(t)$ under the same assumptions on the coefficients as in Section 4. For this purpose, we observe that estimates (4.16) (with $l = 1$) and (4.31) can be rewritten in a simpler and more readable way as

$$(5.1) \quad |(D^j T(t)f)(x)|^p \leq C_{j,p} \frac{e^{\omega_{j,p}t}}{t^{j/2}} (T(t)|f|^p)(x), \quad \forall t > 0, x \in \mathbb{R}^N,$$

for any $p \in (1, +\infty)$, any $j = 1, 2, 3$ and some constants $C_{j,p} > 0$ and $\omega_{j,p} \in \mathbb{R}$.

Theorem 5.1. *Let μ be the invariant measure associated with the semigroup $T(t)$. Moreover, let $l \in \{1, 2, 3\}$ and let assumptions H1-H3, H4-l- γ be satisfied, and fix $p \in (1, +\infty)$. Then, if $\omega_{j,p} \leq 0$,*

$$(5.2) \quad \|D^l T(t)f\|_p \leq C_{l,p} \frac{e^{\omega_{l,p}t}}{t^{l/2}} \|f\|_p, \quad \forall t > 0.$$

If $\omega_{l,p} > 0$, then for any $\omega > 0$ there exist positive constants $\tilde{C}_{l,p} = \tilde{C}_{l,p}(\omega)$ such that

$$(5.3) \quad \|D^l T(t)f\|_p \leq \tilde{C}_{l,p} \frac{e^{\omega t}}{t^{l/2}} \|f\|_p,$$

for any $f \in L^p(\mathbb{R}^N, \mu)$. In particular, if $l = 1$, then $D(A_p) \subset W^{1,p}_\mu(\mathbb{R}^N)$ and there exists a positive constant $M = M(\omega)$ such that

$$(5.4) \quad \|Du\|_p \leq M \|u\|_p^{\frac{1}{2}} \|(A_p - \omega)u\|_p^{\frac{1}{2}}.$$

Proof. We fix $f \in C_b(\mathbb{R}^N)$ and, for any fixed $t > 0$, we integrate (5.1) in \mathbb{R}^N obtaining

$$(5.5) \quad \begin{aligned} \int_{\mathbb{R}^N} |(D^j T(t)f)(x)|^p d\mu(x) &\leq C_{j,p} \frac{e^{\omega_{j,p}t}}{t^{j/2}} \int_{\mathbb{R}^N} (T(t)(|f|^p))(x) d\mu(x) \\ &= C_{j,p} \frac{e^{\omega_{j,p}t}}{t^{j/2}} \int_{\mathbb{R}^N} |f(x)|^p d\mu(x). \end{aligned}$$

By density we can extend (5.5) to all $f \in L^p_\mu$. If $\omega_{l,p} \leq 0$ we are done. In the case when $\omega_{l,p} > 0$, from (5.5) we easily get (5.3) if we recall that $T(t)$ is a semigroup of contractions in L^p_μ and argue as in the last part of the proof of Theorem 3.3.

Finally, we observe that (5.4) follows from (5.2) and (5.3) (with $l = 1$), arguing as in the proof of [4, Corollary 5.4]. \square

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